Degenerations of the Grassmannian $\mathbb{G}(3, 6)$

by

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Contents

Introduction

Let $\mathbb{G}(d,n)$ be the Grassmannian parametrizing d-dimensional linear subspaces in an n-dimensional vector space $\mathcal{V}$. It is a projective scheme embedded in $\mathbb{P}^N$ for $N = \binom{n}{d} - 1$ via its Plücker embedding. Let $\mathcal{L}$ be a distributive lattice. Then one can form the *Hibi variety* $\operatorname{Proj} H_{\mathcal{L}}$, which is a binomial scheme defined by certain relations coming from the lattice $\mathcal{L}$. It is well-known [CHT06] that the Grassmannian $\mathbb{G}(d,n)$ degenerates to a Hibi variety associated to a certain lattice $\mathcal{L}_{d,n}$.

The ideal of the Hibi variety $\operatorname{Proj} H_{\mathcal{L}_{d,n}}$ has a nice initial ideal such that its initial complex is isomorphic to $\mathcal{K} := \Delta_{eq} * \Delta^d$, where Δ_{eq} is a simplicial sphere and Δ^d is a d-simplex. This implies that the Hibi variety degenerates to a Stanley-Reisner scheme $\mathbb{P}(\mathcal{K})$. When $d = 2$, Δ_{eq} is the dual associahedron, and it was shown in [CI11] that in this case $\mathbb{P}(\mathcal{K})$ is unobstructed. The first example where $\mathbb{P}(\Delta_{eq} * \Delta^d)$ is obstructed is for $d = 3, n = 6$, which will be the topic of this thesis.

We first study two special automorphisms of $\mathbb{G}(d, 2d)$ induced by automorphisms of a lattice $\mathcal{L}_{d,2d}$ associated to the Grassmannian $\mathbb{G}(d, 2d)$ and describe these. They generate a subgroup $\mathcal{G} \subset \operatorname{Aut}(\mathbb{G}(d, 2d))$ with $\mathcal{G} = \mathbb{Z}/2 \times \mathbb{Z}/2$. By definition $\mathcal{G}$ acts on the Hibi variety $\operatorname{Proj} H_{\mathcal{L}_{d,2d}}$, and it is also easy to see that it acts on Δ_{eq}. We then compute the cotangent modules T^i $(i = 1, 2)$ for the Stanley-Reisner scheme $\mathbb{P}(\mathcal{K})$. Using a package for the computer algebra software `Macaulay2` [GS, Ilt11], we compute a family of deformations $\mathcal{X} \to \mathcal{T}$ having the Stanley-Reisner scheme $\mathbb{P}(\mathcal{K})$ as its special fiber, the Hibi variety as an intermediate fiber, and the Grassmannian $\mathbb{G}(3, 6)$ as a generic fiber. The group $\mathcal{G}$ acts on $T^1_{A_{\mathcal{K}}}$, and on the base space $\mathcal{T}$. It turns out that the invariant subspace $\mathcal{T}^{\mathcal{G}}$ is smooth of dimension 6.

The last section is devoted to studying the fibers of the family $\mathcal{X} \to \mathcal{T}^{\mathcal{G}}$. In particular we find that there are only three isomorphism classes of irreducible degenerations of $\mathbb{G}(3, 6)$. One of them is the Hibi ring, and the other two are obtained by setting just one of the six deformation parameters to zero. We are able to describe their singular loci.

In Chapter 1 we present preliminary concepts and results. They are stated with the purpose of fixing notation and introducing the uninitiated reader to the terminology.

In Chapter 2 we present the Grassmannian and its Plücker embedding. We discuss its automorphism group, and completely describe the group $\mathcal{G}$ when $d = 2$ and $d = 3$. We give examples for $\mathbb{G}(2, 4)$.

In Chapter 3 we present the necessary background from deformation theory. We give definitions of the cotangent modules $T^i(B/A, M)$ $(i = 0, 1, 2)$ where A and B are rings and M is a B-module. We cite the necessary results of Altmann and Christophersen from [AC10].

In Chapter 4 we define the Hibi ring and explain the construction of the equatorial sphere Δ_{eq}. We explain how in general $\mathbb{G}(d, n)$ degenerates to the Hibi variety and then to the Stanley-Reisner scheme $\mathbb{P}(\mathcal{K})$.

Finally, in Chapter 5 we compute T^i-modules for $i = 1, 2$ using the results of Altmann and Christophersen. We explain how the family $\mathcal{X} \to \mathcal{T}$ was constructed and we analyze its fibers.

There are three appendices. In Appendix A we briefly explain the computational techniques used to obtain primary decompositions of the complicated ideals occuring when studying the family. In Appendix B we include `Macaulay2`-code for computing T^1 and T^2. We also include code for computing a presentation matrix of toric ideals. In Appendix C we include equations of some of the ideals, and an explicit description of the equatorial sphere Δ_{eq}.

Finally, I would like to thank my advisor, Jan Christophersen, for his always open office and his enthusiasm.

Notation and terminology: We will often write := when defining something. The notation $\mathbb{N}$ will always mean the non-negative integers, i.e. the set $\{0, 1, 2, \cdots\}$. The group $\mathrm{PGL}(\mathcal{V})$ is the quotient of $\mathrm{GL}(\mathcal{V})$ by the subgroup of scalar matrices, i.e. scalar multiples of the identity matrix. All rings and modules are commutative, and all rings have an identity element. Fixing a number n, then we denote by $k[\mathbf{x}]$ the polynomial ring $k[x_1, \cdots, x_n]$. A *monomial* in $k[\mathbf{x}]$ is a product $\mathbf{x}^{\mathbf{a}} := x_1^{a_1} \cdots x_n^{a_n}$, where $\mathbf{a} = (a_1, \cdots, a_n) \in \mathbb{N}$. Thus we see that the ring $k[\mathbf{x}]$ is $\mathbb{N}^n$-graded. An ideal I is a *monomial ideal* if it is generated by monomials. We will write $k[\epsilon]$ for $k[x]/(x^2)$. The symbol k will always denote a field, algebraically closed when necessary.

Chapter 1

Preliminaries

This chapter will give a short introduction to the background and notations used in the subsequent chapters.

1.1 Some order theory

Definition 1.1.1. A *partially ordered set* or a *poset* is set P together with a binary relation $\leq$ that is reflexive ($a \leq a$), antisymmetric ($a \leq b$ and $b \leq a$ implie $a = b$) and transitive ($a \leq b$ and $b \leq c$ implies $a \leq c$). If $a, b \in P$ and $a \leq b$ or $b \leq a$, then we say that a and b are *comparable*, otherwise they are *incomparable*. If any two elements are comparable, then P is a *totally ordered set*. ■

All posets considered here will be finite.

Definition 1.1.2. An *order ideal* in a poset $(P, \leq)$ is a possibly empty subset $I \subseteq P$ such that if $a \leq b$ and $b \in I$ then $a \in I$. Denote by $J(P)$ the set of order ideals in P. ■

Example 1.1.3. A poset can be visualized with its *Hasse diagram*. For example, let $X = \{1, 2, 3\}$. If we form the power set $\mathcal{P}(X)$ and let the binary relation be containment $\subseteq$, we obtain a poset which can be visualized as in Figure 1.1. ◊

Definition 1.1.4. A poset $(\mathcal{L}, \leq)$ is a *lattice* if any two $a, b \in \mathcal{L}$ has a *join* $a \vee b$ and a *meet* $a \wedge b$. They are the supremum and the infimum of $\{a, b\}$ with respect to the order $\leq$, respectively. The lattice is *distributive* if the join and meet distribute over each other. ■

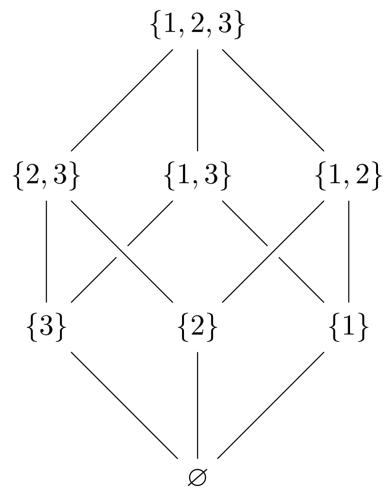

Figure 1.1: The Hasse diagram for $\mathcal{P}(\{1,2,3\})$.

Definition 1.1.5. An element K in a lattice $\mathcal{L}$ is called *join-irreducible* if it is not the minimum of $\mathcal{L}$ and if it cannot be written as $I \vee J$ for $I, J < K$. ■

Example 1.1.3 (continuing from p. 1)**.** The poset $(X, \subseteq)$ is a distributive lattice with join union and meet intersection. The join-irreducible elements are $\{3\}, \{2\}$ and $\{1\}$. This is easily seen from the Hasse diagram in Figure 1.1. ◊

Every finite distributive lattice arises this way:

Theorem 1.1.6 (Birkhoff's representation theorem)**.** *Let $\mathcal{L}$ be a distributive lattice and let P be the poset of join-irreducible elements of $\mathcal{L}$. Then $\mathcal{L}$ is lattice-isomorphic to $J(P)$ with the induced poset structure and join union and meet intersection.*

Proof. See [Bir37]. □

Definition 1.1.7. Let P be a poset. A *chain* (*of length* n) in a P is a sequence $p_1 < p_2 < \cdots < p_n$. A chain is *maximal* if it cannot be extended. A poset is *graded* if every maximal chain has the same length. The *rank* of a graded poset is the length of a maximal chain. ■

For example, the poset in Example 1.1.3 is graded. A grading gives rise to a rank function rank : $P \to \mathbb{N}$. We can define

$$\text{rank}(p) = \sup \Big\{ \text{length of a chain ending at p} \Big\}.$$

Thus, for example, the poset in Figure 1.1 has rank 3.

1.2 Simplicial complexes and Stanley-Reisner rings

A Stanley-Reisner ring is a quotient of a polynomial ring by a square-free monomial ideal. These ideals are described geometrically in terms of finite simplicial complexes.

Definition 1.2.1. An (abstract) *simplicial complex* Δ on the *vertex set* $[n] = \{1, \ldots, n\}$ is a collection of subsets of the vertex set. The elements of Δ are called *faces*, and they are closed under taking subsets: if $F \in \Delta$ and $f \subseteq F$, then $f \in \Delta$. A face $F \in \Delta$ of cardinality $i+1$ has *dimension* i and is called an *i-face* of Δ. The *dimension* $\dim(\Delta)$ of Δ is $\max_{F \in \Delta} \dim F$. The *full simplex* Δ^d is the simplicial complex associated to the power set of the vertex set $[d]$. A simplicial complex is *pure* if all maximal faces have the same dimension. ■

Note that a simplicial complex is determined by the set of its maximal faces.

Definition 1.2.2. If P is a poset, then the *order complex* $\Delta(P)$ of P is the simplicial complex with vertices the elements of P and finite chains of elements of P as faces. Note that $\Delta(P)$ is pure if and only if P is graded. ■

Definition 1.2.3. The *order polytope* $\mathcal{O}(P)$ of a poset P is the convex hull of $\{\chi_I : I \in J(P)\} \subset \mathbb{R}^{\#P}$, where χ_I is the characteristic vector of I, i.e. $\chi_I(p) = 1$ if $p \in I$ and $\chi_I(p) = 0$ otherwise. ■

Example 1.2.4. Let Δ be the simplicial complex with maximal faces $\{1, 2\}$, $\{2, 3\}$ and $\{1, 3\}$. We see that, as topological spaces, $\Delta \approx S^1$. ◊

We define some natural operations on simplicial complexes:

Definition 1.2.5. Let $f \in \mathcal{K}$. Then the *link at f in $\mathcal{K}$* is the set

$$\mathrm{link}(f, \mathcal{K}) := \{g \in \mathcal{K} \mid g \cap f = \varnothing \text{ and } f \cup g \in \mathcal{K}\}.$$

If $\mathcal{G}$ is any other simplicial complex, then the *join* of $\mathcal{K}$ and $\mathcal{G}$ is the complex defined by

$$\mathcal{K} * \mathcal{G} := \{f \vee g | f \in \mathcal{K}, g \in \mathcal{G}\},$$

where $\vee$ means disjoint union. If $g \subseteq [n]$, denote by $\bar{g} := 2^g$ the full simplex on g. Then we define $\partial g := \bar{g} \backslash \{g\}$ as the *boundary* of g. ■

For the category theory oriented reader, note that $\mathcal{K} * \mathcal{G}$ is the category theoretic product of $\mathcal{K}$ and $\mathcal{G}$.

Every simplicial complex $\mathcal{K}$ has a *geometric realization*, denoted by $|\mathcal{K}|$. It is defined as

$$|\mathcal{K}| = \Big\{\alpha\colon [n] \to [0,1] \,\Big|\, \{i \mid \alpha(i) \neq 0\} \in \mathcal{K} \text{ and } \sum_{i=1}^{n} \alpha(i) = 1\Big\}.$$

Example 1.2.6. If Δ_1, Δ_2 are two intervals, that is, two-vertex complexes, then their join is a tetrahedron. We have $\partial\Delta_1 \approx S^0$ as topological spaces. ♢

Example 1.2.4 (continuing from p. 3)**.** If $F = \{1\}$, then $\mathrm{link}_\Delta(F) = \{2,3\}$, the disjoint union of the two other vertices. In general, if Δ is a triangulated n-sphere S^n, and f is any vertex of Δ, then $\mathrm{link}_\Delta(f)$ is a triangulated $(n-1)$-sphere S^{n-1}.

If Γ has maximal faces $\{0\}$ and $\{4\}$, then $\Delta * \Gamma$ has maximal faces $\{0,1,2\}, \{0,2,3\}, \{0,1,3\}, \{1,2,4\}, \{2,3,4\}$ and $\{1,3,4\}$. It is a triangulated 2-sphere, so $\Delta * \Gamma \approx S^2$. ♢

Definition 1.2.7. If f is an r-dimensional face of $\mathcal{K}$, the *valency* of f, $v(f)$, is defined to be number of $(r+1)$-dimensional faces containing f. Thus $v(f)$ equals the number of vertices in $\mathrm{link}(f, \mathcal{K})$. ■

We will occasionally use some notation for special simplicial complexes. Write $\Sigma\mathcal{K}$ for the suspension of the complex $\mathcal{K}$. Note that $\Sigma\mathcal{K} = \mathcal{K} * \{1,2\}$. Write E_n for the boundary of the n-gon.

Now some algebra. Let $k[\mathbf{x}] := k[x_1, \ldots, x_n]$, where k is a field. Simplicial complexes determine squarefree monomials in the following way: A subset $\sigma \subseteq [n]$ give a squarefree vector in $\{0,1\}^n$, which has a 1 in the i'th spot when $i \in \sigma$ and a 0 otherwise. This allows us to write $\mathbf{x}^\sigma = \prod_{i\in\sigma} x^i$.

Definition 1.2.8. Let $\mathcal{K}$ be a simplicial complex. Its *Stanley-Reisner ideal* is the squarefree monomial ideal

$$I_{\mathcal{K}} = \big\langle \mathbf{x}^\sigma \,\big|\, \sigma \notin \mathcal{K} \big\rangle \subseteq k[\mathbf{x}]$$

generated by the nonfaces of Δ. The *Stanley-Reisner ring* of Δ is the quotient ring $A_{\mathcal{K}} := k[\mathbf{x}]/I_{\mathcal{K}}$. ■

Note that if $\mathcal{K} = \Delta_1 * \Delta_2$, then $A_{\mathcal{K}} = A_{\Delta_1} \otimes_k A_{\Delta_2}$.

Example 1.2.4 (continuing from p. 3)**.** The simplicial complex Δ give rise to the Stanley-Reisner ideal $(x_1x_2x_3)$ in $k[x_1, x_2, x_3]$. ♢

We associate to Stanley-Reisner rings $A_{\mathcal{K}}$ the schemes $\mathbb{A}(\mathcal{K}) = \operatorname{Spec} A_{\mathcal{K}}$ and $\mathbb{P}(\mathcal{K}) = \operatorname{Proj} A_{\mathcal{K}}$. The latter looks like the complex $\mathcal{K}$ – its simplices have just been replaced by projective spaces intersecting in the same way as the corresponding faces of Δ:

Theorem 1.2.9. *The correspondence $\Delta \mapsto I_\Delta$ is a bijection from simplicial complexes on $[n]$ to squarefree monomial ideals in $k[\mathbf{x}]$. More precisely, let $\mathfrak{m}^\tau$ denote the ideal $\langle x_i \mid i \in \tau \rangle$, where $\tau \subset [n]$. Then*

$$I_\Delta = \bigcap_{\sigma \in \Delta} \mathfrak{m}^{\bar{\sigma}},$$

where $\bar{\sigma} = \{1, \dots, n\} \backslash \sigma$, is the complement of σ in $[n]$.

Proof. See the first chapter of [MS05]. □

Example 1.2.4 (continuing from p. 3)**.** The Stanley-Reisner scheme $\mathbb{P}(\Delta)$ is the union of three projective lines. ♢

For more on Stanley-Reisner rings, see [Sta96].

1.3 Initial ideals and Gröbner bases

We fix some notation and definitions about Gröbner bases. For more details, see for example [Eis95, Chapter 15].

We can identify monomials in $k[\mathbf{x}]$ with points in $\mathbb{N}^n$. A total order $<$ on $\mathbb{N}^n$ is a *term order* if the zero vector $\mathbf{0}$ is the unique minimal element and if $a < b$ implies $a + c < b + c$ for all $a, b, c \in \mathbb{N}^n$.

Given a term order on $\mathbb{N}^n$, every polynomial $f \in k[\mathbf{x}]$ has an *inital monomial*, denoted $in_<(f)$: it is defined as the highest term of f in the total order on $k[\mathbf{x}]$ induced by the order on $\mathbb{N}^n$. If I is an ideal of $k[\mathbf{x}]$, then its *initial ideal* is the monomial ideal

$$in_<(I) := \langle in_<(f) \mid f \in I \rangle$$

generated by the initial terms.

Definition 1.3.1. Let I be an ideal in $k[\mathbf{x}]$ and $<$ a term order. We say that $\{f_1, \dots, f_r\}$ is a *Gröbner basis* for I if

$$in_<(I) = \langle in_<(f_1), \dots, in_<(f_r) \rangle$$

Note that a Gröbner basis is automatically a generating set for the ideal. ■

A Gröbner basis is *minimal* if no monomial $in_<(f_i)$ is redundant, and *reduced* if for any two f_i, f_j, no term of f_j is divisible by $in_<(f_i)$. The monomials which do not lie in $in_<(I)$ are called the *standard monomials*.

Given a set of generators for an ideal I, there is an algorithm for computing a Gröbner basis of I, called the *Buchberger algorithm*. For more on this, see [Eis95] and the first chapter of [Stu96].

One also has the notion of an order by a *weight vector*. Fix $\omega = (\omega_1, \cdots, \omega_n) \in \mathbb{R}^n$. For any polynomial

$$f = \sum c_i \mathbf{x}^{\mathbf{a}_i}$$

we define the *initial form* $in_\omega(f)$ to be the sum of all terms $c_i \mathbf{x}^{\mathbf{a}_i}$ such that the inner product $\omega \cdot \mathbf{a}_i$ is maximal. For any ideal I we define the *initial ideal* (with respect to ω) to be the ideal generated by the initial forms:

$$in_\omega(I) := \langle in_\omega(f) \mid f \in I \rangle.$$

If ω is chosen sufficiently generic, the initial ideal is monomial.

Fixing I and a term order $<$, there is always a weight vector ω representing $<$:

Proposition 1.3.2. *For any term order $<$ and any ideal $I \subset k[\mathbf{x}]$, there exists a non-negative integer weight vector $\omega \in \mathbb{N}^n$ such that*

$$in_\omega(I) = in_<(I).$$

Proof. See [Stu96, Proposition 1.11] or [Eis95, Proposition 15.16] □

The process of passing to the initial ideal is a flat deformation. This is proved, for example, in [Eis95, Theorem 15.17]. The precise result takes the following form. Set $P := k[\mathbf{x}]$ and let $P[t]$ be a polynomial extension of P in one variable. For any $g \in P$, define $\tilde{g}$ as follows. Write $g = \sum c_i \mathbf{x}^{\mathbf{a}_i}$ as a sum of monomials where $c_i \in k^*$. Let $b = \max_i \omega \cdot \mathbf{a}_i$ and set

$$\tilde{g} = t^b g(t^{-\omega_i} x_1, \cdots, t^{-\omega_n} x_n)$$

Because of the way $\tilde{g}$ is defined, one sees that $\tilde{g}$ is $in_\omega(g)$ plus terms involving t. For any ideal I, let $\tilde{I}$ be the ideal of $P[t]$ generated by $\{\tilde{g} \mid g \in I\}$. It follows $P[t]/(t, \tilde{I}) = P/in_\omega(I)$.

In fact we have:

Theorem 1.3.3. *For any ideal $I \subset P$, the $k[t]$-algebra $P[t]/\tilde{I}$ is flat as a $k[t]$-module. Furthermore*

$$P[t]/\tilde{I} \otimes_{k[t]} k[t, t^{-1}] = P/I[t, t^{-1}]$$

and

$$P[t]/\tilde{I} \otimes_{k[t]} k[t]/(t) = P/in_\omega(I).$$

Using the language of deformation theory, this says that there is a family of deformations $\mathcal{X} \to \operatorname{Spec} k[t]$ such that the special fiber is $\operatorname{Spec} P/in_\omega(I)$ and the generic fiber is $\operatorname{Spec} P/I$, where $\mathcal{X} = \operatorname{Spec} P[t]/\tilde{I}$, and all fibers except the special fiber are isomorphic.

1.4 Toric ideals and triangulations

In this section we will introduce toric varieties as presented in [Stu96].

Let $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ be a finite subset of $\mathbb{Z}^d$. By abuse of notation, we will also denote by $\mathcal{A}$ the $d \times n$-matrix with columns the coordinates of the elements of $\mathcal{A}$. We call $\mathcal{A}$ a *point configuration.*

The point configuration $\mathcal{A}$ induces a semigroup homomorphism

$$\pi \colon \mathbb{N} \to \mathbb{Z}^d, \ \mathbf{u} = (u_1, \ldots, u_n) \mapsto \sum_i u_i \mathbf{a}_i.$$

The image of π is the semigroup

$$\mathbb{N}\mathcal{A} = \sum_i \mathbb{N}\mathbf{a}_i.$$

The map π lifts to a homomorphism of semigroup algebras:

$$\hat{\pi} \colon k[\mathbf{x}] \to k[\mathbf{t}^{\pm 1}], \ x_i \mapsto \mathbf{t}^{\mathbf{a}_i}.$$

The kernel of $\hat{\pi}$ is the *toric ideal* $I_\mathcal{A}$. We will call any ideal obtained in this way from a point configuration a *toric ideal.* This differs from the terminology in, for example, [Ful93], in that we do not require toric ideals to be normal. $I_\mathcal{A}$ is clearly a prime ideal.

We write $\mathbb{Z}\mathcal{A}$ for the sublattice of $\mathbb{Z}^n$ spanned by $\mathcal{A}$. The *dimension* of $\mathcal{A}$ is defined as the dimension of $\mathbb{Z}\mathcal{A}$. We have the following:

Lemma 1.4.1. *The Krull dimension of the residue ring* $k[\mathbf{x}]/I_\mathcal{A}$ *is* $\dim(\mathcal{A})$.

Proof. This is Lemma 4.2 in [Stu96]. □

Every vector $\mathbf{u} \in \mathbb{Z}^n$ can be written uniquely as a difference $\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^-$ where $\mathbf{u}^+, \mathbf{u}^- \in \mathbb{N}^n$. Denote by $\ker \pi$ the sublattice of $\mathbb{Z}^n$ consisting of all vectors $\mathbf{u}$ such that $\pi(\mathbf{u}+) = \pi(\mathbf{u}^-)$.

The *cone* spanned by $\mathcal{A}$ is the set

$$\operatorname{cone}(\mathcal{A}) := \Big\{ \sum_i c_i \mathbf{a}_i \mid \mathbf{a}_i \in \mathcal{A}, c_i \in \mathbb{R}_{\geq 0} \Big\}.$$

We have $\operatorname{cone}(\mathcal{A}) = \mathbb{N}\mathcal{A} \otimes_{\mathbb{Z}} \mathbb{R}$. A *fan* is a finite collection of cones such that each face of each cone is also in the collection, and such that any pair of cones in the collection intersects in a common face. A fan is *simplicial* if the generators of each cone are linearly dependent over $\mathbb{R}$.

If $<$ is any term order and $I \subset k[\mathbf{x}]$ is any ideal, then $in_<(I)$ is a monomial ideal. We can associate to I a simplicial complex $\Delta_<(I)$. It is called the *initial complex* of I (with respect to $<$) and is defined as the simplicial complex whose Stanley-Reisner ideal is the radical of $in_<(I)$.

Definition 1.4.2. If σ is a subset of $\mathcal{A}$, then write $\operatorname{cone}(\sigma)$ for the cone spanned by σ. A *triangulation* of $\mathcal{A}$ is a collection Δ of subsets of $\mathcal{A}$ such that the set

$$\Big\{ \operatorname{cone}(\sigma) \mid \sigma \in \Delta \Big\}$$

is the set of cones in a simplicial fan whose support equals $\operatorname{cone}(\mathcal{A})$. Note that as a set, a triangulation is a simplicial complex. ■

If $\mathcal{A} = \{\mathbf{a}_1, \cdots, \mathbf{a}_n\}$, identify the set $\mathcal{A}$ with the index set $\{1, \cdots, n\}$. Every sufficiently generic vector $\omega \in \mathbb{R}^n$ defines a triangulation Δ_ω as follows: A subset $\{i_1, \cdots, i_r\}$ is a face of Δ_ω if there is a vector $\mathbf{c} = (c_1, \cdots, c_d) \in \mathbb{R}^d$ such that

$$\begin{aligned} &\mathbf{a}_j \cdot \mathbf{c} = \omega_j \text{ if } j \in \{i_1, \ldots, i_r\} \text{ and} \\ &\mathbf{a}_j \cdot \mathbf{c} < \omega_j \text{ if } j \in \{1, \ldots, n\} \backslash \{i_1, \ldots, i_r\}. \end{aligned}$$

Definition 1.4.3. A triangulation Δ of $\mathcal{A}$ is *regular* if $\Delta = \Delta_\omega$ for some $\omega \in \mathbb{R}^n$. ■

Sturmfels shows in [Stu96] the following important theorem:

Theorem 1.4.4 (Sturmfels)**.** *Regular triangulations correspond to initial complexes of the toric ideal $I_{\mathcal{A}}$. More precisely, if $\omega \in \mathbb{N}^n$ represents $<$ for $I_{\mathcal{A}}$, then $\Delta_<(I_{\mathcal{A}}) = \Delta_\omega$.*

A triangulation is *unimodular* if $vol(\sigma) = 1$ for every maximal simplex $\sigma \in \Delta$. Here $vol(\sigma)$ denotes the normalized volume. This translates into the ideal $I_{\mathcal{A}}$ being squarefree:

Proposition 1.4.5. *The initial ideal $in_<(I_{\mathcal{A}})$ is square-free if and only if the corresponding regular triangulation $\Delta_<$ of $\mathcal{A}$ is unimodular.*

Proof. This is Corollary 8.9 in [Stu96]. □

1.5 SAGBI bases

Let $\mathcal{F} = \{f_1, \ldots, f_n\}$ be a set of polynomials in $k[\mathbf{t}] = k[t_1, \ldots, t_d]$ and let $R = k[\mathcal{F}]$ be the sub-algebra they generate. Fix a term order $<$ on $k[\mathbf{t}]$. The *initial algebra* $in_<(R)$ is the k-vector space spanned by the monomials $\{in_<(f) \mid f \in R\}$. A *canonical basis* or a *SAGBI basis*[1] is a finite subset $\mathcal{C}$ of R such that $in_<(R)$ is generated as a k-algebra by the set of monomials $\{in_<(f) \mid f \in \mathcal{C}\}$.

Not all algebras possess canonical bases as the finiteness condition is quite strong. For example, Sturmfels shows in an example in [Stu96] that the invariant ring of the alternating group A_3 has no finite canonical basis.

Suppose $in_<(f_i) = \mathbf{t}^{\mathbf{a}_i}$, and let $\mathcal{A} \subset \mathbb{N}^d$ be the set $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$. Let $k[\mathbf{x}] = k[x_1, \ldots, x_n]$ and consider the k-algebra map from $k[\mathbf{x}]$ onto $k[\mathcal{F}] \subseteq k[\mathbf{t}]$ defined by $x_i \mapsto f_i$ and let I be its kernel. Similarly, consider the map defined by $x_i \mapsto in_<(f_i)$. The kernel of this map is the toric ideal $I_{\mathcal{A}}$.

Now, let $\omega \in \mathbb{R}^d$ be any weight vector representing the term order $<$ for the polynomials in $\mathcal{F}$. If we consider $\mathcal{A}$ as a $d \times n$-matrix with transpose $\mathcal{A}^T$, then $\mathcal{A}^T\omega$ is a vector in $\mathbb{R}^n$, which can be used as a weight vector on $k[\mathbf{x}]$.

Theorem 1.5.1. *Suppose $\mathcal{F}$ is a canonical basis for the subalgebra it generates. Then*

1. *every reduced Gröbner basis $\mathcal{G}$ of $I_{\mathcal{A}}$ lifs to a reduced Gröbner basis $\mathcal{H}$ of I, i.e. the elements of $\mathcal{G}$ are the initial forms (with respect to $\mathcal{A}^T\omega$) of the elements of $\mathcal{H}$, and*

2. *every regular triangulation of $\mathcal{A}$ is an initial complex of the ideal I.*

Proof. This is Corollary 11.6 in [Stu96]. □

In geometric terms, this says that every parametrically presented projective variety possessing a SAGBI basis deforms to a projective toric variety.

The theorem can be translated to a theorem in algebraic geometry. Let $k[\mathcal{F}]$ be a finitely generated homogeneous k-algebra possessing a finite SAGBI basis. A presentation $k[\mathbf{x}] = k[x_1, \ldots, x_n] \to k[\mathcal{F}]$ gives an embedding $\operatorname{Proj} k[\mathcal{F}] \to \mathbb{P}^{n-1}$. Let $k[in_<(\mathcal{F})]$ denote the algebra of initial forms of $\mathcal{F}$ and let $\mathcal{A}$ denote the corresponding point configuration. Then the theorem takes the following form:

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[1]The acronym "SAGBI" stands for "sub-algebra analog for Gröbner bases of ideals".

Theorem 1.5.2. *There exists a one-parameter family of embedded deformations η having* $\operatorname{Proj} k[\mathcal{F}]$ *as generic fiber and the toric variety* $\operatorname{Proj} k[\mathbf{x}]/I_{\mathcal{A}}$ *as special fiber.*

$$\eta : \quad \begin{array}{ccccc} \operatorname{Proj} k[\mathbf{x}]/I_{\mathcal{A}} & \longrightarrow & \mathcal{X} & \hookrightarrow & \operatorname{Spec} k[t] \times \mathbb{P}^{n-1} \\ \big\downarrow & \ulcorner & \big\downarrow {\scriptstyle \pi_{|\mathcal{X}}} & \swarrow {\scriptstyle \pi} & \\ \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} k[t] & & \end{array}$$

Here π is flat.

Chapter 2

The Grassmannian

In this chapter we introduce the Grassmannian and study its automorphism group. In particular we study a group $\mathcal{G}$ of automorphisms coming from a certain distributive lattice. This group will be important later on.

2.1 Definition

First, fix an n-dimensional vector space $\mathcal{V}$ over the algebraically closed field k. Let $\mathbb{G}(d, \mathcal{V})$ be the Grassmannian of d-dimensional linear subspaces of $\mathcal{V}$.

Note that to give a d-dimensional subspace of $\mathcal{V}$ is equivalent to giving a $(d-1)$-dimensional subspace of the projective space $\mathbb{P}(\mathcal{V}) = \mathbb{P}^{n-1}$. Some authors use the notation $\mathbb{G}(d, \mathcal{V})$ to mean the collection of d-dimensional *projective* subspaces (for example [Har95]). For us, the notation will always refer to the set of d-dimensional *linear* subspaces of $\mathcal{V}$.

We will often refer to a d-dimensional linear subspace as a *d-plane* to save space. When using coordinates, one often uses the notation $\mathbb{G}(d, n)$ instead of $\mathbb{G}(d, \mathcal{V})$.

2.2 Projective structure

To fix notation, we describe the projective structure of the Grassmannian. First choose some basis of $\mathcal{V}$. Let $M = (x_{ij})$ be a generic $d \times n$-matrix, so that its row span is an element of $\mathbb{G}(d, n)$. Denote by $[n] = \{1, \ldots, n\}$ the set of positive integers less than or equal to n. If $I \subseteq [n]$ is a subset of cardinality d, denote by M_I the submatrix of M using the columns determined by I.

We have the following result:

Lemma 2.2.1. *Let M be a $d \times n$ matrix. The set $\{\det M_I\}_{\#I=d}$ of maximal minors of M determines the row span of M uniquely. More precisely, a matrix M' has the same row span as M if and only if there exists some non-zero constant c such that $\det M_I = c \det M'_I$ for all maximal minors* $\det M_I$.

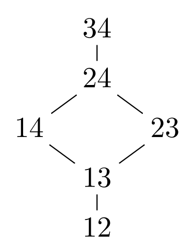
Proof. See [MS05, Chapter 14]. □

We can thus use the $N+1$ minors $\{\det M_I\}_{\#I=d}$ as projective coordinates on the Grassmannian, where $N = \binom{n}{d} - 1$. Ordering them lexicographically, we can represent a point $W \in \mathbb{G}(d,n)$ by $[\dots, \det M_I, \dots] \in \mathbb{P}^N$. These coordinates are called the *Plücker coordinates* on $\mathbb{P}^N$.

The association of a matrix to its list of maximal minors determines a closed embedding $\mathbb{G}(d,n) \to \mathbb{P}^N$ in the following way: Let $k[I] := k[\dots, I, \dots]$ be the polynomial ring with variables indexed by the subsets of $[n]$ of cardinality d, and let $k[\dots, x_{ij}, \dots]$ be the polynomial ring with variables indexed by the entries of a generic $d \times n$-matrix X. Then one defines a map $k[I] \to k[x_{ij}]$ by $I \mapsto \det X_I$. The kernel of this map is known as the ideal of *Plücker relations*, or just the *Plücker ideal*. For example, if $d = 2$ and $n = 4$, the Plücker ideal is generated by the single quadratic homogeneous equation

$$[14][23] - [13][24] + [12][34].$$

We want to describe a Gröbner basis for the Plücker ideal. To do this, it is convenient to introduce a poset $\mathcal{P}$ as follows. Let $\mathcal{P}$ be the poset whose underlying set is the set of subsets of $[n]$ of cardinality d. Then define $I \leq_{\mathcal{P}} J$ if $I_i \leq J_i$ for $i = 1, \dots, d$. Note that $\mathcal{P}$ has a natural structure as a distributive lattice: If $I = [i_1 \dots i_d]$ and $J = [j_1 \dots j_d]$, then we have $I \vee J = [\max(i_1, j_1), \dots, \max(i_d, j_d)]$ and $I \wedge J = [\min(i_1, j_1), \dots, \min(i_d, j_d)]$. When thinking of it as a distributive lattice, we will denote it by $\mathcal{L}_{d,n}$. For example, if $d = 2$ and $n = 4$, the poset $\mathcal{P} = \mathcal{L}_{2,4}$ have the form:



(2.1)

The lattice when $d = 3$ and $n = 6$ is included at the end of this chapter as Figure 2.1. Note that when $n = 2d$, the associated distributive lattice has a natural horizontal and vertical symmetry.

It is well-known that the ideal of Plücker relations is generated by homogeneous quadrics: Totally order the maximal minors lexicographically, and call this order $\preccurlyeq$ (so that it is a linear extension of $\leq_{\mathcal{P}}$). Also denote by $\preccurlyeq$ the reverse lexicographic term order on $k[I]$ induced by the variable ordering $\preccurlyeq$.

Theorem 2.2.2. *The ideal I of Plücker relations has a Gröbner basis under $\preccurlyeq$ consisting of homogeneous quadrics. More precisely, the products IJ of incomparable pairs in the poset $\mathcal{P}$ generate the initial ideal $\mathrm{in}_{\preccurlyeq}(I)$.*

Proof. This is proved for example in [MS05]. For a classical proof and an explicit description of the relations, see the very readable article by Kleiman and Laksov [KL72]. □

Example 2.2.3. Consider $\mathbb{G}(2,4)$. A matrix representing a 2-plane is a 2×4-matrix. The maximal minors are ordered as

$$[12], [13], [14], [23], [24], [34]$$

under $\preccurlyeq$. Thus the points of $\mathbb{G}(2,4)$ in the Plücker embedding are precisely the points

$$[x_{11}x_{22} - x_{12}x_{21} : x_{11}x_{23} - x_{13}x_{21} : \cdots : x_{13}x_{24} - x_{14}x_{23}] \in \mathbb{P}^6.$$

It is easy to recover a plane W from the Plücker coordinates and conversely. For example, let W be the 2-plane that is the row span of the 2×4-matrix below:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}.$$

Then the Plücker coordinates of W are $[1 : 2 : 3 : 1 : 2 : 1]$. The matrix can be recovered by first assuming that the submatrix $M_{[12]}$ is the identity matrix (this is possible since this minor is non-zero), and then succesively solve linear equations. ◊

The homogeneous coordinate ring of $\mathbb{G}(d,n)$ is thus the sub k-algebra of $k[x_{ij}]$ generated by the maximal minors of a generic $d \times n$-matrix. It is well-known that the minors form a SAGBI basis for this sub-algebra. See for example [Stu93].

The dimension of $\mathbb{G}(d,n)$ is easily computed: To give a d-plane in $\mathcal{V}$ is equivalent to giving a $d \times n$-matrix, but this is only unique up to left-multiplication by a $d \times d$-matrix. Hence $\dim \mathbb{G}(d,n) = dn - d^2 = d(n-d)$. In particular, $\mathbb{G}(3,6) = 3 \cdot 3 = 9$.

2.3 Automorphism group

We want to know about the automorphism group of $\mathbb{G}(d, \mathcal{V})$.

First we fix some terminology: Let $\mathrm{Aut}(X)$ denote the set of all automorphisms of X. If $X \subset Y$ then $\mathrm{Aut}(X, Y)$ is the subgroup

$$\left\{\varphi \in \mathrm{Aut}(Y) \,\middle|\, \varphi(X) = X\right\}$$

of automorphisms of Y fixing X.

The results presented in this section motivate the choice of invariant family used in the last chapter. The first result we will prove is that every automorphism of the Grassmannian is projective. To prove this, we need a lemma:

Lemma 2.3.1. *The Picard group of the Grassmannian* $\mathbb{G}(d, \mathcal{V})$ *is isomorphic to* $\mathbb{Z}$.

Proof. See [Ful97, Chapter 9.2]. □

We were unable to find a proof of the next proposition in the literature, so we include a proof for completeness.

Proposition 2.3.2. *We have*

$$\mathrm{Aut}\left(\mathbb{G}(d, \mathcal{V})\right) = \mathrm{Aut}\left(\mathbb{G}(d, \mathcal{V}), \mathbb{P}^N\right).$$

Proof. The embedding $\iota\colon \mathbb{G}(d, \mathcal{V}) \to \mathbb{P}^N$ provides a line bundle $\mathcal{L}$ such that $\mathcal{L} \simeq \iota^*\mathcal{O}_{\mathbb{P}^N}(1)$. It is generated by its global sections, which are the determinants of the d-minors of a generic $d \times n$-matrix. Any automorphism φ of $\mathbb{G}(d, \mathcal{V})$ induces an automorphism of $\mathrm{Pic}\ \mathbb{G}(d, \mathcal{V}) = \mathbb{Z}$, so a generator of $\mathrm{Pic}\ \mathbb{G}(d, \mathcal{V})$ must be sent to another generator. Clearly, $\varphi^*\mathcal{L} = \mathcal{L}$, since $\mathcal{L}^\vee$ has no global sections. This means that φ induces an isomorphism of k-vector spaces:

$$\varphi^*\colon \Gamma(\mathbb{G}(d, \mathcal{V}), \mathcal{L}) \to \Gamma(\mathbb{G}(d, \mathcal{V}), \mathcal{L}).$$

But the d-minors are k-linearly independent (this follows since they form a SAGBI basis), and so this isomorphim lifts uniquely to an isomorphism of $\Gamma(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))$, and this in turn induces an automorphism of $\mathbb{P}^N$. It is well-known that every automorphism of $\mathbb{P}^N$ is of this form. □

Remark. *This proof is just a minor modification of Example 7.1.1 in [Har77], where he proves that the automorphisms of* $\mathbb{P}^n$ *are given by* $PGL(n)$.

We will give a description of the automorphism group of the Grassmannian.

Theorem 2.3.3 (Chow). *If $2d \neq n$, then*

$$\operatorname{Aut}\big(\mathbb{G}(d,n)\big) = \operatorname{PGL}(\mathcal{V}).$$

If $2d = n$, then

$$\operatorname{Aut}\big(\mathbb{G}(d,n)\big) = \mathbb{Z}/2 \times \operatorname{PGL}(\mathcal{V}),$$

where $\mathcal{V}$ is a vector space of dimension n.

Remark. *This was originally proved by Chow in 1949, in his paper* "On the geometry of algebraic homogeneous spaces"*. A more modern treatment was given in, for example, the paper* "Automorphisms of Grassmannians" *by Cowen. See [Cho49] or [Cow89].*

The theorem says that every automorphism of the Grassmannian is induced by an automorphism of $\mathcal{V}$ if $2d \neq n$. If however $2d = n$, then there is one additional automorphism coming from a duality map. We will quicly describe it.

We introduce some notation. We want to define a map

$$*\colon \mathbb{G}(d,\mathcal{V}) \to \mathbb{G}(n-d,\mathcal{V}).$$

To do this, we need to identify $\mathcal{V}$ with its dual $\mathcal{V}^*$: Choose a basis $\{e_1,\dots,e_n\}$ of $\mathcal{V}$ and let $\{\delta_1,\dots,\delta_n\}$ be the dual basis. Then we define $\iota\colon \mathcal{V} \to \mathcal{V}^*$ by $e_i \mapsto \delta_i$. If $j\colon \mathcal{V} \to \mathcal{V}^{**}$ is the natural isomorphism, we have $j \circ \iota = \iota^t$.

For a linear subspace $W \subset \mathcal{V}$, let $W^\perp$ denote the annihilator of W: it is the set of linear functionals that vanish on W:

$$W^\perp := \Big\{\lambda \in \mathcal{V}^* \,\Big|\, \lambda(w) = 0 \text{ for all } w \in W\Big\}.$$

Then we define the map $*\colon \mathbb{G}(d,\mathcal{V}) \to \mathbb{G}(n-d,\mathcal{V})$ by $*(W) = \iota^{-1}(W^\perp)$. We call the map $*$ *the duality map* (relative to the identification $\mathcal{V} \simeq \mathcal{V}^*$).

We give a sketch proof of Theorem 2.3.3.

Sketch proof of 2.3.3. Let V be a vector subspace of dimension $d+1$. Then one defines the Schubert cycle

$$\sigma(V) = \Big\{W \in \mathbb{G}(d,\mathcal{V}) \,\Big|\, W \subseteq V\Big\}.$$

Similarly, let V' be a vector subspace of dimension $d-1$. Then one defines the Schubert cycle

$$\Sigma(V') = \Big\{ W \in \mathbb{G}(d,\mathcal{V}) \,\Big|\, W \supseteq V' \Big\}.$$

The proof goes like this: One shows that any automorphism of the Grassmannian must either preserve or reverse Schubert cycles, meaning that if φ is an automorphism of $\mathbb{G}(d,\mathcal{V})$, then either $\varphi(\sigma(V)) = \sigma(\tilde{V})$ for some $d+1$-dimensional $\tilde{V}$, or $\varphi(\sigma(V)) = \Sigma(\tilde{V})$ for some $d-1$ dimensional $\tilde{V}$. For dimensional reasons, only one of these options can occur if $2d \neq n$. If $2d = n$, both can occur, so the duality isomorphism is allowed. Finally, one shows that a Schubert cycle-preserving automorphism must come from an automorphism of the n-dimensional vector space $\mathcal{V}$. □

Lemma 2.3.4. *The map* $*\colon \mathbb{G}(d,n) \to \mathbb{G}(n-d,n)$ *is induced by the isomorphism*

$$\bigwedge^{d} \mathcal{V} \to \bigwedge^{n-d} \mathcal{V}$$

given by sending a basis vector e_I *to* $\epsilon_{IJ} e_J$ *where* ϵIJ *is such that* $e_I \wedge e_J = \epsilon_{IJ} e_1 \wedge \cdots \wedge e_n$.

Example 2.3.5 (Two-planes in four-space)**.** In this case, we compute that the duality map is given by

$$\big[e_{12} : e_{13} : e_{14} : e_{23} : e_{24} : e_{34}\big] \mapsto \big[e_{34} : -e_{24} : e_{23} : e_{14} : -e_{13} : e_{12}\big].$$

Let V be the 2-plane given by the 2×4-matrix

$$\begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & 2 & 4 & 6 \end{pmatrix}.$$

Then its Plücker coordinates are given by $P = [1 : 2 : 3 : 1 : 2 : 1]$ and its image under the duality map is $*P = [1 : -2 : 1 : 3 : -2 : 1]$. This corresponds to the matrix

$$\begin{pmatrix} 1 & -2 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{pmatrix},$$

which is easily seen to be orthogonal to the original matrix. ◊

2.4 Automorphisms coming from the lattice $\mathcal{L}_{d,2d}$

When $n = 2d$, there are two obvious lattice isomorphisms of $\mathcal{L}_{d,2d}$. One can turn the lattice up-side down, and one can mirror it vertically. We name these two automorphisms υ and λ, respectively (the *upsilon* for "up" and the *lambda* for "left"). They induce obvious automorphisms of $\mathbb{P}^N = \mathbb{P}(\wedge^d \mathcal{V})$.

Consider for example the distributive lattice associated to $\mathbb{G}(2,4)$, as seen in Equation 2.1. The automorphism λ is given by exchanging [14] and [23], and leaving the other variables fixed. The automorphism υ is given similarly by turning the lattice upside down.

Example 2.4.1. Let P be the plane given by the row span of the matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}.$$

The Plücker coordinates are given by $[1:2:3:1:2:1] \in \mathbb{P}^5$. Its image under λ is then $[1:2:1:3:2:1]$. This corresponds to the plane given by the matrix

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 5 & 11 & 7 & 1 \end{pmatrix}.$$

If we apply the duality map $*\colon \mathbb{G}(2,4) \to \mathbb{G}(2,4)$, we obtain the plane given by the row span of the matrix

$$\begin{pmatrix} 4 & -3 & 2 & -1 \\ 8 & -7 & 6 & 5 \end{pmatrix}.$$

If we let M be the 4×4 matrix

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

one sees that for the plane P, we have $* \circ \lambda(P) = M(P)$, where $M(P)$ is the image of the plane P under the linear transformation corresponding to M. ♢

This last observation holds for all planes in $\mathbb{G}(2,4)$, namely that $* \circ \lambda = M$. It is proven by direct computation. In the rest of this section we will show that the analogous statement holds also for $\mathbb{G}(3,6)$.

Lemma 2.4.2. *Let $\mathcal{V}$ be a 6-dimensional vector space. The automorphism $\lambda \circ * \in \mathrm{PGL}\left(\wedge^3 \mathcal{V}\right)$ is induced by the matrix*

$$\varphi = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathrm{PGL}(\mathcal{V}).$$

That is, we in formulas, we have $\wedge^3\varphi = \lambda \circ *$.

Proof. This is again direct computation. Using `Macaulay2`, one calculates that the matrix of $\lambda \circ *$ equals the matrix of $\wedge^3\varphi$. □

Lemma 2.4.3. *Let $\mathcal{V}$ be a 6-dimensional vector space. The automorphism $\upsilon \in \mathrm{PGL}(\wedge^3\mathcal{V})$ is induced by the matrix*

$$\psi = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathrm{PGL}(\mathcal{V}).$$

That is, in formulas, we have $\wedge^3\psi = \upsilon$.

Proof. Direct computation. □

Proposition 2.4.4. *Let $\mathcal{V}$ be a 6-dimensional vector space. The automorphisms υ and λ of $\mathbb{P}(\wedge^3\mathcal{V})$ induce automorphisms $\upsilon, \lambda \in \mathrm{Aut}\left(\mathbb{G}(3,6)\right)$, and λ is not induced by an automorphism of $\mathcal{V}$.*

Proof. Every matrix $\chi \in \mathrm{GL}(\mathcal{V})$ induces an automorphism of $\mathbb{G}(3,6)$ by right-multiplication of a matrix representing the row space of an element in $\mathbb{G}(3,6)$. Above it was shown that υ was induced by the invertible matrix ψ, so it must induce an automorphism of $\mathbb{G}(3,6)$.

Since we know that $*$ is an automorphism of $\mathbb{G}(3,6)$, and that $* \circ \lambda$ is induced by the matrix φ, it follows that $* \circ \lambda$ is an automorphism of $\mathbb{G}(3,6)$. Applying $* = *^{-1}$ on the left implies that λ is an automorphism of $\mathbb{G}(3,6)$.

Finally, λ cannot be the image of a matrix in $\mathrm{GL}(\mathcal{V})$, since from the description of $\mathrm{Aut}(\mathbb{G}(d,2d))$ as $\mathbb{Z}/2 \times \mathrm{PGL}(\mathcal{V})$, it follows that $\lambda = (1,\varphi)$, where the second factor represents automorphisms that are images of automorphisms in $\mathrm{GL}(\mathcal{V})$. □

The automorphisms λ and υ generate a subgroup $\mathcal{G}$ of $\mathrm{Aut}(\mathbb{G}(d,\mathcal{V}))$ isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$.

I have not been able to prove these results for all $\mathbb{G}(d, 2d)$, but the matrices that show up have such symmetric shapes that one should be very surprised if this does not hold generally.

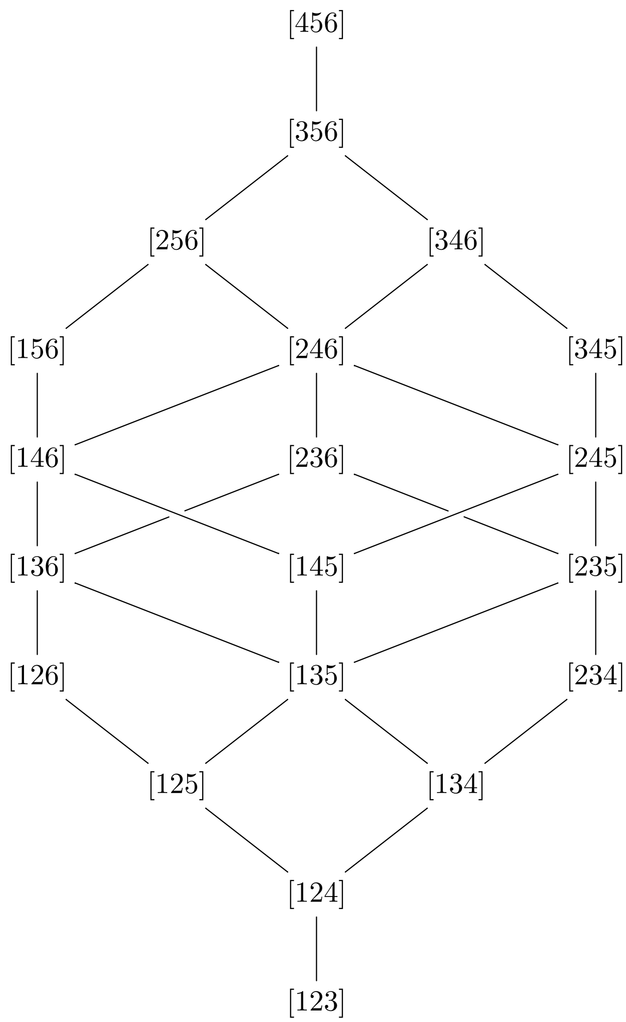


Figure 2.1: The distributive lattice $\mathcal{L}_{3,6}$.

Chapter 3

Deformation theory

This chapter gives a quick overview of the techniques of deformation theory used in this thesis. Our main sources are [Har10] and [Ser06].

3.1 Deformation theory

Let X be a scheme over an algebraically closed field k. Deformation theory studies how X varies in a flat family. Recall that a flat family is a flat morphism of schemes $\mathcal{X} \to S$. A *deformation of X over S* is just a flat family $\mathcal{X} \to S$ such that S has a distinguished point $0 \in S$, and such that the the fiber over 0 is X. Thus a deformation of X is equivalent to giving a cartesian square η:

$$\eta : \quad \begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & \ulcorner & \downarrow{\scriptstyle \pi} \\ \operatorname{Spec} k & \longrightarrow & S, \end{array}$$

where π is flat. We call S the *parameter space* and $\mathcal{X}$ the *total space* of the family. If S is the spectrum of an Artinian ring, then we call η an *infinetesimal deformation*. If $S = \operatorname{Spec} k[\epsilon]$, then we call η a *first-order deformation*. We call $\pi^{-1}(0) = X$ the *special fiber*.

If $X \hookrightarrow \mathbb{P}^n$ is a closed embedding, one defines similarly an *embedded deformation* η to be a cartesian commutative diagram:

$$\eta : \quad \begin{array}{ccccc} X & \longrightarrow & \mathcal{X} & \hookrightarrow & \mathbb{P}^n \times S \\ \downarrow & \ulcorner & \downarrow{\scriptstyle \pi_{|\mathcal{X}}} & \swarrow{\scriptstyle \pi} & \\ \operatorname{Spec} k & \longrightarrow & S & & \end{array}$$

If all the closed points of S have the same residue field k, it follows that every fiber of $\mathcal{X} \to S$ is a subscheme of $\mathbb{P}^n$. Since π is flat, each fibre over S has the same Hilbert polynomial $P(t)$, under the additional hypothesis that S is integral and noetherian (see for example Theorem 9.9, Chapter 2 in [Har77]).

Theorem 3.1.1 (Existence of the Hilbert Scheme)**.** *Let Y be a closed subscheme of $\mathbb{P}^n$. Then there exists a projective scheme $\mathcal{H}$, the* Hilbert scheme*, parametrizing closed subschemes of $\mathbb{P}^n$ with the same Hilbert polynomial $P(t)$ as Y, and there exists a universal subscheme $\mathcal{X} \subset \mathbb{P}^n \times \mathcal{H}$, flat over $\mathcal{H}$, such that the fibers of W over closed points of $\mathcal{H}$ are all closed subschemes of $\mathbb{P}^n$ with the same Hilbert polynomial $P(t)$.*

Furthermore, $\mathcal{H}$ is universal: If S is a scheme and $\mathbb{P}^n \times S \supset \mathcal{Y} \to S$ is a family, all of whose fibers have the same Hilbert polynomial $P(t)$, there is a unique morphism $S \to \mathcal{H}$, such that

$$\mathcal{Y} = S \times_{\mathcal{H}} \mathcal{X} \subset \mathbb{P}^n \times S.$$

Proof. A proof can be found in [Ser06, Chapter 4.3]. □

Definition 3.1.2. For any subscheme Y of a scheme X, one can form the *normal sheaf*

$$\mathcal{N}_{Y/X} := \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y),$$

where $\mathcal{I}$ is the ideal sheaf on Y in X. ■

It is known that there is a 1–1 correspondence between embedded deformations of $Y \subseteq X$ over the dual numbers and global sections of the normal sheaf $\mathcal{N}_{Y/X}$.

If $X = \mathbb{P}^{n-1}$, and Y is a closed subscheme with Hilbert polynomial $P(t)$, we can think of Y as a point on the Hilbert scheme $\mathcal{H}$ parametrizing subschemes of $\mathbb{P}^{n-1}$ with Hilbert polynomial $P(t)$. Then it is easily seen that $\mathcal{N}_{Y/\mathbb{P}^{n-1}}$ is naturally isomorphic to the Zariski tangent space of $\mathcal{H}$ at the point Y. Thus if Y corresponds to a non-singular point on $\mathcal{H}$, the dimension of $\mathcal{H}$ can be computed as the dimension of $\mathcal{N}_{Y/\mathbb{P}^{n-1}}$.

Note that the Grassmannian $\mathbb{G}(d, n)$ is the Hilbert scheme parametrizing subvarieties with Hilbert polynomial $P(t) = \binom{t+d-1}{d-1}$. The following example gives a high-tech way to compute the its dimension.

Example 3.1.3. A d-plane W in an n-dimensional vector space $\mathcal{V}$ becomes after projectivization a $(d-1)$-plane in $\mathbb{P}^{n-1}$. It is the complete intersection of $n - d$ hyperplanes, so that we have a surjection

$$\bigoplus_{i=1}^{n-d} \mathcal{O}_W(-1) \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow 0$$

of locally free sheaves of the same rank. This implies that this is an isomorphism, so that we have equalities

$$\begin{aligned}\mathcal{N}_{W/\mathbb{P}^{n-1}} &= \mathcal{H}om(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_W) \\ &= \mathcal{H}om(\bigoplus_{i=1}^{n-d} \mathcal{O}_W(-1), \mathcal{O}_W) = \bigoplus_{i=1}^{n-d} \mathcal{O}_W(1)\end{aligned}$$

Thus $h^0(\mathcal{N}_{W/\mathbb{P}^{n-1}}) = d(n-d)$, as expected, since that is the dimension of the Grassmannian as computed in Chapter 2. $\diamondsuit$

3.2 The T^i-functors

Let A be a ring. For an A-algebra B, we may form the *cotangent complex*, and take its homology to form certain T^i functors. We will briefly introduce these functors. For details, see for example [Har10, Chapter 3].

Let $R = A[\mathbf{x}]$ be a polynomial ring surjecting onto B with kernel I, so that we have an exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow B \longrightarrow 0.$$

Now choose a free R-module F presenting I, and let Q be the module of relations, so that we have an exact sequence:

$$0 \longrightarrow Q \longrightarrow F \xrightarrow{j} I \longrightarrow 0.$$

Let F_0 be the submodule of F defined by all *Koszul relations*, namely the relations of the form $j(a)b - j(b)a$ for $a, b \in F$. Since $j(F_0) = 0$, we have that F_0 is a submodule of Q.

Having defined these modules, we can define the *cotangent complex*:

$$\mathbb{L}_* : \qquad L_2 \xrightarrow{d_2} L_1 \xrightarrow{d_1} L_0$$

Let $L_2 = Q/F_0$, $L_1 = F \otimes_R B = F/IF$, and let $L_0 = \Omega_{R/A} \otimes_R B$. Let d_2 be the map induced by the inclusion $Q \to F$ and let d_1 be the map induced by the universal derivation $d : R \to \Omega_{R/A}$. Then one checks that $\mathbb{L}_*$ really is a complex, and that it is well-defined.

If M is any B-module, we can form the complex $\hom_B(\mathbb{L}_*, M)$. Taking homology, one obtains, by definition, the T^i-modules:

$$T^i(B/A, M) := h^i(\mathrm{Hom}_B(\mathbb{L}_*, M)),$$

where h^i is the homology functor.

Let $M = B$. Then we have the following identifications:

- $T^0(B/A, B) = \mathrm{Hom}_B(\Omega_{B/A}, B) = \mathrm{Der}_A(B, B)$, the *tangent module* of B over A.
- $T^1(B/A, B) = \mathrm{coker}\big(\mathrm{Hom}_B(\Omega_{R/A}, B) \to \mathrm{Hom}_B(I/I^2, B)\big)$.
- $T^2(B/A, B) = \mathrm{Hom}_B(Q/F_0, B)/\mathrm{im} d_2^\vee$.

We will often just write T^i_B when $M = B$. It is known that $T^1(B/k, B)$ classifies first-order deformations of $\mathrm{Spec}\, B$. Lets compute a toy example.

Example 3.2.1. Let $B = k[x, y]/(xy)$ be the Stanley-Reisner ring associated to the simplicial complex $\partial\Delta^1$. We want to compute $T^i(B/k, B)$ for $i = 0, 1, 2$.

In the construction above, let $R = k[x, y]$. The the ideal (xy) is principal, so the module of relations is zero. Thus $T^2(B/k, B) = 0$ since it is a quotient of a zero module.

Again, since I/I^2 is principal, we have an identification $\mathrm{Hom}_B(I/I^2, B) \simeq B$. Since $\Omega_{R/k} \otimes_R B$ is generated by dx and dy, the dual is generated by $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. The map d_2 sends a combination $f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}$ to $fy + gx$, so that the image of d_2 is the ideal (x, y). Hence $T^1(B/k, B) = B/(x, y) = k$. This means that all first-order deformations of B looks like $k[x, y][t]/(xy - t)$. ♢

This construction may be globalized to schemes. That is, given a morphism $f\colon X \to Y$ of schemes and a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, we get $\mathcal{O}_X$-modules $\mathcal{T}^i(X/Y, \mathcal{F})$ for $i = 0, 1, 2$.

3.3 Obstruction calculus

Given a set of first-order deformations of a projective scheme X, there is an algorithm for lifting these to higher order. More precisely, given a deformation family $\mathcal{X} \to \mathcal{T}$, where $\mathcal{T} = \mathrm{Proj}\, k[t_1, \ldots, t_n]/(t_1, \ldots, t_n)^2$, one wants to lift this deformation to higher and higher powers of the maximal ideal.

We briefly describe an algorithm to do this, the *Massey product algorithm*. We will follow the exposition in [Ilt11]. First, fix some notation. Let $X = \operatorname{Proj} B = \operatorname{Proj} S/I$ be a projective scheme, where S is a polynomial ring and I as an ideal. Consider a free resolution of S/I:

$$\cdots \longrightarrow S^l \xrightarrow{R^0} S^m \xrightarrow{F^0} S \longrightarrow S/I \longrightarrow 0$$

Let $\phi_i \in \operatorname{Hom}(S^m/\operatorname{im}R^0, S)$ $(i = 1, \ldots, t)$ represent a subset of a basis for $T^1(B/k, B)$. Introduce deformation parameters $t_1, \ldots, t_t$, and let $\mathfrak{m} = \langle t_1, \cdots, t_t \rangle$ be the ideal generated by the deformation parameters. Consider the map $F^1 : S[\mathtt{t}]^m \to S[\mathtt{t}]$ given by

$$F^1 = F^0 + \sum_{i=1}^{t} t_i \phi_i.$$

It follows that there is a map $R^1 : S[\mathtt{t}]^l \to S[\mathtt{t}]^m$ with $R^1 \equiv R^0 \pmod{\mathfrak{m}}$ satisfying the first order deformation equation

$$F^1 R^1 \equiv 0 \pmod{\mathfrak{m}^2}.$$

The problem is to lift this solution modulo higher and higher powers of $\mathfrak{m}$. In general, there are obstructions to doing this, and they are found in the d-dimensional vector space $T^2(B/k, B)$. For more on this, see for example [Har10, Chapter 10].

What one can do instead, is try to solve the augmented deformation equation

$$(F^i R^i)^T + C^{i-2} G^{i-2} \equiv 0 \pmod{\mathfrak{m}^{i+1}}, \tag{3.1}$$

where $(F^i R^i)^T$ denotes the transpose of $F^i R^i$. Here, the matrices $G^{i-2} \colon S[\mathtt{t}] \to S[\mathtt{t}]^d$ and $C^{i-2} \colon S[\mathtt{t}]^d \to S[\mathtt{t}]^l$ are congruent modulo $\mathfrak{m}^i$ to G^{i-3} and C^{i-3}, respectively. Furthermore, G^i and C^i vanish for $i < 0$, and C^0 is of the form VD, where $V \in \operatorname{Hom}(S^d, S^l)$ gives representatives for a basis of $T^2(B/k, B)$ and $D \in \operatorname{Hom}(S^d, S^d)$ is a diagonal matrix.

Given a solution $(F^i, R^i, G^{i-2}, C^{i-2})$ of (3.1), one wants to lift the solution to work modulo $\mathfrak{m}^{i+2}$. One solves first for F^{i+1} and G^{i+1} by working modulo $I + \operatorname{im}(G^{i-2})^T + \mathfrak{m}^{i+2}$. Having found these, one can solve for R^{i+1} and C^{i-1}. This is exactly what the `Macaulay2` package `VersalDef` does.

The matrices G^i now give equations for the base space of the lifted deformation for higher and higher powers of $\mathfrak{m}^i$. In nice cases, the lifting stops, meaning that equation (3.1) is true not merely modulo $\mathfrak{m}^{i+1}$, but over $S[\mathtt{t}]$. If we had used all deformation parameters, this would have been a *versal* family for X. Note that if we were to hope for a versal family to exist, a necessary condition is that $T^1(B/k, B)$ is finite-dimensional over k.

3.4 Deformation theory of Stanley-Reisner schemes

In the papers [AC04, AC10] Altmann and Christophersen describe how to calculate $T^1(B/k, B)$ and $T^2(B/k, B)$ for Stanley-Reisner schemes purely in terms of the combinatorics of the simplicial complexes. We briefly restate the main results, and refer to their articles for details.

Let $\mathcal{K}$ be a simplicial complex with vertices $[n]$. Let $A_{\mathcal{K}}$ denote the coordinate ring of the Stanley-Reisner scheme associated to $\mathcal{K}$. It has a natural $\mathbb{Z}^n$-grading: $\mathbf{x}^{\mathbf{a}}$ has degree $\mathbf{a}$. The $A_{\mathcal{K}}$-modules T^i inherit this grading, so that we have decompositions

$$T^i_{A_{\mathcal{K}}} = \bigoplus_{\mathbf{c}\in\mathbb{Z}^n} T^i_{A_{\mathcal{K}},\mathbf{c}}.$$

Every $\mathbf{c} \in \mathbb{Z}^n$ can be decomposed into $\mathbf{a} - \mathbf{b}$ with $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$. It will conventient to write degrees as a fraction of variables: the expression $\Pi_i x_i^{a_i}/\Pi x_j^{b_j}$ will mean the degree $\mathbf{a} - \mathbf{b}$ where $\mathbf{a} = (a_i)$ and $\mathbf{b} = (b_i)$. The *support* of $\mathbf{a} = (a_i)$ is $a := \{i \in [n] \mid a_i \neq 0\}$.

Theorem 3.4.1. ([AC04, Theorem 13]) *The homogeneous pieces in degree* $\mathbf{c} = \mathbf{a} - \mathbf{b}$ *(with disjoint supports a and b) of the cotangent cohomology of the Stanley-Reisner ring $A_{\mathcal{K}}$ vanish unless $a \in \mathcal{K}$, $\mathbf{b} \in \{0,1\}^{n+1}$, $b \subseteq [\mathrm{link}(a)]$ and $b \neq \varnothing$.*

This says $T^i_{\mathbf{c}}(\mathcal{K})$ depends only on the supports a and b. Therefore we will often denote it simply by $T^i_{a-b}(\mathcal{K})$. The computations may be reduced to the case $a = \varnothing$ by the following lemma:

Proposition 3.4.2. ([AC04, Proposition 11]) *If $b \subseteq [\mathrm{link}(a)]$, then the map $f \mapsto f\backslash a$ induces isomorphisms $T^i_{\varnothing-b}(\mathrm{link}(a, \mathcal{K})) \cong T^i_{a-b}(\mathcal{K})$ for $i = 1, 2$.*

Definition 3.4.3. Define $\mathcal{B}(\mathcal{K})$ to be the set of $b \subseteq [\mathcal{K}]$, $|b| \geq 2$, with the properties

1. $\mathcal{K} = L * \partial b$ where L is a $(n - |b| + 1)$-sphere if $b \notin \mathcal{K}$
2. $\mathcal{K} = L * \partial b \cup \partial L * \bar{b}$ where L is a $(n - |b| + 1)$-ball if $b \in \mathcal{K}$.

Note that if $\mathcal{K}$ is not a sphere, then $\mathcal{B}(\mathcal{K}) = \varnothing$. ∎

We need to recall some definitions from PL-topology: a *combinatorial n-sphere* is a simplicial complex $\mathcal{K}$ such that $|\mathcal{K}|$ is PL-homeomorphic to $|\partial\Delta_{n+1}|$. A simplicial complex $\mathcal{K}$ of dimension n is a *combinatorial n-manifold* if for all non-empty faces $f \in K$, $|\,\mathrm{link}(f, \mathcal{K})|$ is a combinatorial sphere of dimension $n - \dim f - 1$. In dimension less than four, all triangulations of topological manifolds are combinatorial manifolds. For details, see for example [Hud69].

Theorem 3.4.4. *If $\mathcal{K}$ is a combinatorial manifold and $\mathbf{c} = \mathbf{a} - \mathbf{b}$ then*

$$\dim_k T^1_{A_\mathcal{K},\mathbf{c}} = \begin{cases} 1 & \text{if } a \in \mathcal{K} \text{ and } b \in \mathcal{B}(lk(a,\mathcal{K})), \\ 0 & \text{otherwise.} \end{cases}$$

A basis for $T^1_{A_\mathcal{K}}$ may be explicitly described: if $\phi \in T^1_{A_\mathcal{K}} \neq 0$ and $x_p \in I_\mathcal{K}$, then $\phi(x_p) = x^{\mathbf{a}} x_{p \setminus b}$ if $b \subseteq p$ and 0 otherwise.

In [AC10] there is a table of simplicial complexes $\mathcal{K}$ with $\dim \mathcal{K} \leq 2$ and $\mathcal{B}(\mathcal{K}) \neq \varnothing$ together with the cardinality of $\mathcal{B}(\mathcal{K})$. The table is reproduced in Chapter 5, Table 5.2.

The results for computing $T^2_{A_\mathcal{K}}$ are not as precise. We state a combination of Proposition 4.8 in [AC10] and Lemma 4.2 in [CI11], where we assume that $|\mathcal{K}|$ is a sphere and that $\mathcal{K}$ is a flag complex. Define $L_b = \cap_{b' \subset b} \operatorname{link}(b', \mathcal{K})$.

Proposition 3.4.5. *If $\mathcal{K}$ is a simplicial flag complex such that $|\mathcal{K}| \approx S^n$, then $T^2_{\varnothing - b} = 0$ unless $\partial b \subset \mathcal{K}$. If $\partial b \subset \mathcal{K}$, then $T^2_{\varnothing - b}$ may be computed as follows:*

- *If $b \in \mathcal{K}$, then $T^2_{\varnothing - b} = 0$.*
- *If $b \notin \mathcal{K}$, then $T^2_{\varnothing - b} \simeq \tilde{H}^0(|K| \backslash |\partial b * L_b|, k) \simeq \tilde{H}_{n-|b|}(L_b, k)$.*

This is true even when the degree $n - |b| = -1$ with the convention that $\tilde{H}_{-1}(\varnothing) = k$.

We will use these results to compute $T^i_{A_\mathcal{K}}$ $(i = 1, 2)$ for an actual example in Chapter 5.

Chapter 4

Degenerations of $\mathbb{G}(d, n)$

In this chapter we describe how in general the Grassmannian $\mathbb{G}(d, n)$ degenerates: first to a toric variety, then to a Stanley-Reisner scheme. The chapter follows the exposition of [CHT06] closely.

4.1 The Hibi ring

We first define for any distributive lattice $\mathcal{L}$ a projective toric variety Proj $H_{\mathcal{L}}$.

Let $\mathcal{L}$ be a distributive lattice and let $k[\mathcal{L}]$ be the polynomial ring whose variables are the elements of $\mathcal{L}$. For each pair of elements $I, J \in \mathcal{L}$, define the Hibi relation

$$IJ - (I \wedge J)(I \vee J).$$

The *Hibi ideal* (or the *lattice ideal*) $I_{\mathcal{L}}$ is the ideal generated by the Hibi relations. Note that if I and J are comparable, then the Hibi relation vanishes, so we need only consider incomparable elements. The *Hibi ring* is the k-algebra $H_{\mathcal{L}} = k[\mathcal{L}]/I_{\mathcal{L}}$.

Takayuki Hibi proved in [Hib87] the following theorem:

Theorem 4.1.1 (Hibi). *If $\mathcal{L}$ is a distributive lattice, then*

- *the Hibi ring $H_{\mathcal{L}}$ is a toric, normal, Cohen-Macaulay algebra with a straightening law,*
- *the ideal $I_{\mathcal{L}}$ has a quadratic squarefree initial ideal whose associated simplicial complex is the chain complex of $\mathcal{L}$, and*
- *$H_{\mathcal{L}}$ is Gorenstein if and only if the poset of join-irreducibe elements of $\mathcal{L}$ is graded.*

Let P be the poset of join-irreducible elements in $\mathcal{L}$ such that $\mathcal{L}$ is isomorphic to $J(P)$ as distributive lattices, and let $\mathcal{O}(P)$ be the order polytope of P, that is, the convex hull of the characteristic vectors. Let $M(\mathcal{O}(P)) = \text{cone}(\{1\} \times \mathcal{O}(P))$ be the cone over the polytope. Then Birkhoff's theorem implies that $H_{\mathcal{L}} = k[M(\mathcal{O}(P))]$:

Proposition 4.1.2. *$H(\mathcal{L})$ is isomorphic to the semigroup ring $k[M(\mathcal{O}(P))]$.*

We have of course already seen distributive lattices. See Equation 2.1 and Figure 2.1 in Chapter 2. For example, in the distributive lattice associated to $\mathbb{G}(2, 4)$, the only incomparable elements are [14] and [23], and so the Hibi ideal is just generated by the single binomial $[14][23] - [13][24]$. This implies in particular that the Hibi variety, $\operatorname{Proj} H_{\mathcal{L}}$ is a cone over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$.

In fact, this is always the case. If $\mathcal{L}_{d,n}$ is the lattice associated to a Grassmannian $\mathbb{G}(d, n)$, then the minimum and the maximum of the lattice never occur in the Hibi relations, so that $\operatorname{Proj} H_{\mathcal{L}_{d,n}}$ is always the cone over a toric variety.

4.2 The equatorial sphere

We describe the equatorial sphere of Reiner and Welker, as presented in their paper [RW05]. Througout this section, let P be any graded poset having n elements and of rank r.

Reiner and Welker give for every graded poset P a special triangulation of the order polytope $\mathcal{O}(P)$. The triangulation has several pleasant properties of which we list two:

- It is a unimodular triangulation.
- It is isomorphic, as an abstract simplicial complex, to the join of an r-simplex with a $(\#P - r - 1)$-sphere, which we will denote by Δ_{eq}. This is the *equatorial sphere.*

Definition 4.2.1. A chain of order ideals $I_1 \subset I_2 \subset \ldots \subset I_t$ is called *equatorial* if $f := \sum \chi_{I_i}$ satisfies $\min_{p \in P} f(p) = 0$ and for every $j \in [2, r]$, there exists a covering relation $p_{j-1} < p_j$ with p_{j-1} of rank $j-1$ and p_j of rank j such that $f(p_{j-1}) = f(p)$. ■

Definition 4.2.2. A chain of order ideals $I_1 \subset I_2 \subset \ldots \subset I_t$ is called *rank-constant* if it is constant along ranks, i.e. if $f(p) = f(q)$ whenever p and q are elements of the same rank in P. ■

Definition 4.2.3. The *equatorial complex* Δ_{eq} is the subcomplex of the order complex $\Delta(J(P))$ whose faces are indexed by equatorial chains of order ideals. ■

Reiner and Welker proves in [RW05] the following:

Proposition 4.2.4. *The collection of all cones*

$$conv(\chi_I : I \in \mathcal{R} \cup \mathcal{E}),$$

where $\mathcal{R}$ (resp. $\mathcal{E}$) is a chain of non-empty rank-constant (resp. equatorial) ideal in P, gives a regular unimodular triangulation of $\mathcal{O}(P)$.

We call the above triangulation the *equatorial triangulation* of $\mathcal{O}(P)$. The proposition implies that it is abstractly isomorphic to $\Delta_{eq} * \Delta^d$ (this is Corollary 3.8 in [RW05]).

Example 4.2.5. Consider the lattice $\mathcal{L}_{2,4}$ associated to $\mathbb{G}(2, 4)$. Then one computes that $\Delta_{eq} = \partial\Delta^1$, is the two-point simplicial complex $\{[14], [23]\}$, so that $\Delta_{eq} \approx S^0$. ♢

Example 4.2.6. Consider $\mathbb{G}(2, 5)$. The one computes that Δ_{eq} is a pentagon. ♢

Example 4.2.7. Consider $\mathbb{G}(3, 6)$. The poset P of join-irreducible element of $\mathcal{L}_{3,6}$ is shown in Figure 4.1. The poset P has rank 5 and cardinality 9. It follows from the Reiner-Welker construction that P has a triangulation isomorphic to $\Delta_{eq} * \Delta^5$, where Δ_{eq} is a $9 - 5 - 1 = 3$-sphere.

In $J(P) = \mathcal{L}_{3,6}$ in Figure 2.1, the rank-constant elements are [123], [124], [135], [246], [356] and [456]. ♢

Recall that in Chapter 2 we studied an automorphism subgroup $\mathcal{G}$ of $\mathrm{Aut}(\mathbb{G}(d, 2d))$. In this case, the lattices $\mathcal{L}_{d,2d}$ are horizontally and vertically symmetric, and the rank-constant element lie on the vertical axis. It follows that the action of $\mathcal{G}$ on $\mathcal{L}_{d,2d}$ induces an action on Δ_{eq}. This will be used in the next chapter.

Notice also that the action sends Hibi-relations into Hibi-relations, so that we have an action on the Hibi ring $H_{\mathcal{L}_{d,2d}}$ also.

4.3 The degenerations of $\mathbb{G}(d, n)$

In Chapter 2 we stated that the homogeneous coordinate ring of the Grassmannian is the k-algebra generated by the $d \times d$-minors of a generic $d \times n$-matrix. Let $<$ be any *diagonal* term order, meaning that the main diagonal

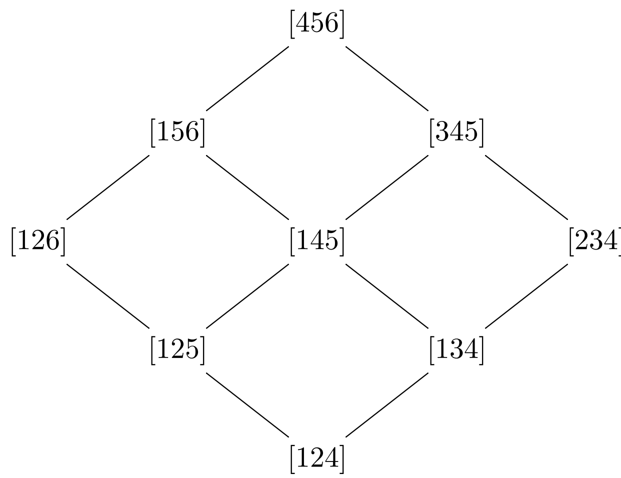


Figure 4.1: The poset of join-irreducible elements of $\mathcal{L}_{3,6}$.

term is the initial term for each $d \times d$-minor. The set of maximal minors is a SAGBI basis for this algebra under the term order $<$.

The SAGBI property of the $d\times d$-minors implies immediately from Proposition 1.5.1 that there is a degeneration of $\operatorname{Proj} k[\det X_I] = \mathbb{G}(d,n)$ to the toric variety $\operatorname{Proj} k[in_<(\det X_I)]$.

Lemma 4.3.1. *The semigroup ring $k[in_<(\det X_I)]$ is isomorphic to the Hibi ring $H_{\mathcal{L}_{d,n}}$.*

Proof. By counting, one sees that

$$in_<(\det X_I) in_<(\det X_J) = in_<(\det X_{I\vee J}) in_<(\det X_{I\wedge J}),$$

so that we have a surjective map onto the Hibi ring. The map must be an isomorphism since both rings are integral domains of the same dimension. □

Now, from Proposition 4.2.4, we know that the polytope defining the Hibi ring has a regular unimodular triangulation, corresponding to a simplicial complex $\mathcal{K} = \Delta_{eq} * \Delta^d$. Regular triangulations of polytopes correspond to initial ideals of toric ideals from Theorem 1.4.4, and it follows that the Hibi ring degenerates to the Stanley-Reisner ring $A_{\mathcal{K}} = k[I]/I_{\mathcal{K}}$.

We sum this up in a theorem:

Theorem 4.3.2. *Let $\mathbb{G}(d, n) \hookrightarrow \mathbb{P}^N$ be the Grassmannian in its Plücker embedding. Then there exists a flat family $\mathcal{X} \to \mathcal{S}$ of embedded deformations having the Grassmannian as generic fiber and the Hibi variety* $\operatorname{Proj} H_{\mathcal{L}_{d,n}}$ *as a fiber at a closed point. The special fiber is the Stanley-Reisner scheme* $\operatorname{Proj} A_{\mathcal{K}}$ *where* $\mathcal{K} = \Delta_{eq} * \Delta^d$.

This implies of course that every invariant of the Grassmannian in the Plücker embedding that is invariant under deformation is shared by the Hibi ring and the Stanley-Reisner scheme. In particular, they all have the same dimension, degree and are all Gorenstein. The last property follows since we know that the central fiber $\operatorname{Proj} A_{\mathcal{K}}$ is Gorenstein.

Chapter 5

Degeneration of $\mathbb{G}(3, 6)$

Using the theory of the previous chapters, we construct a specific example of a deformation having the Stanley-Reisner scheme $\mathbb{P}(\Delta_{eq} * \Delta^5)$ as the special fiber, and the Grassmannian $\mathbb{G}(3, 6)$ as generic fiber.

5.1 The equatorial sphere

Using a computer program, we constructed the equatorial sphere Δ_{eq} of Reiner and Welker. It has 14 vertices and 42 maximal faces. The f-vector was computed to be $(14, 56, 84, 42)$. We will denote it by $\mathcal{S}$.

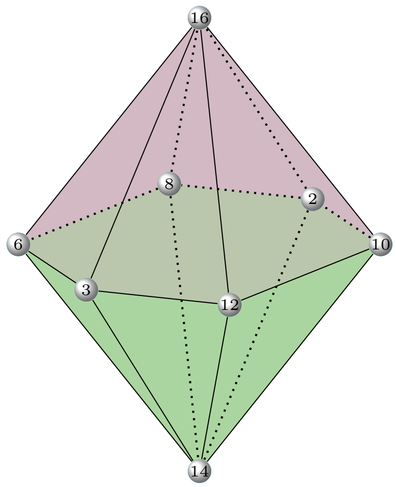
Its automorphism group was calculated using the computer algebra software SAGE [S+12], and it is isomorphic to $\mathbb{Z}/2 \times D_4$. The vertices come in three orbits, of size $8, 4, 2$, respectively. The size two orbit corresponds to the action of flipping the distributive lattice $\mathcal{L}_{3,6}$ up-side down.

One way to describe a three-dimensional simplicial sphere is by its links at vertices, which are two-dimensional spheres. The links come in three isomorphism classes. They are described in Figure 5.1.

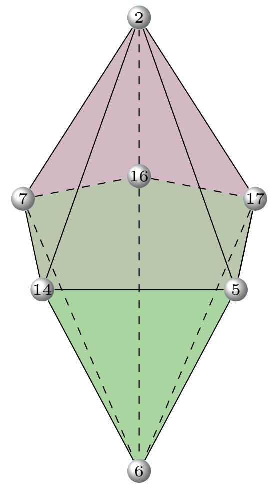
The links at other vertices can be obtained by applying automorphisms. We have written down the orbits of the drawn links in Table 5.1.

5.2 Calculation of $T^i_{A_\mathcal{S}}$ $i = 1, 2$

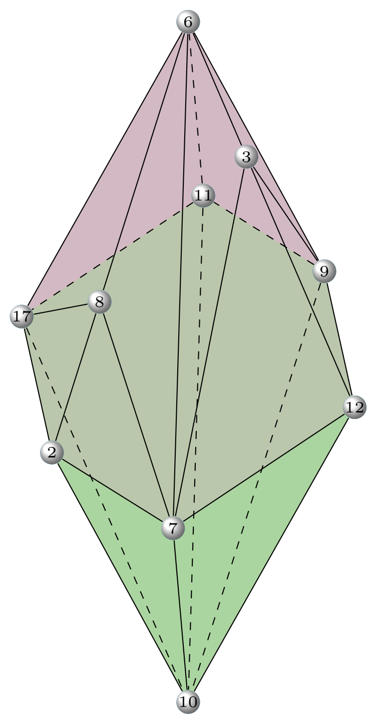
In this section we will calculate a basis in degree zero and less of the modules $T^1_{A_\mathcal{S}}$ $(i = 1, 2)$ using the results of [AC10]. We first consider $T^1_{A_\mathcal{S}}$.



(a) Link at 7.



(b) Link at 8.



(c) Link at 16.

Figure 5.1: The three isomorphism classes of links at vertices.

Link at 16	7	3	12	9	11	17	2	8	10	6
Link at 14	7	3	12	15	11	5	2	8	10	6
Link at 10	11	5	17	2	7	12	9	15	16	14
Link at 6	11	5	17	8	7	3	9	15	16	14
Link at 7	6	3	12	10	2	8	14	16		
Link at 11	5	17	16	9	15	14	10	6		
Link at 8	6	3	12	10	2	8	14			
Link at 15	6	3	12	10	11	14	9			
Link at 3	16	9	15	14	7	6	12			
Link at 17	6	8	2	10	11	16	5			
Link at 2	14	5	17	16	7	10	8			
Link at 5	6	8	2	10	11	14	17			
Link at 9	6	3	12	10	11	16	15			
Link at 12	16	9	15	14	7	10	3			

Table 5.1: All links isomorphic to the link at 10.

5.2.1 The first-order deformations

Using Theorem 3.4.4 and Table 5.2, one can find an explicit basis for $T^1_{A_{\mathcal{S}},\mathbf{c}}$ for each graded piece $\mathbf{c}$.

$\mathcal{K}$	$\mathcal{B}$	$\lvert\mathcal{B}(\mathcal{K})\rvert$
$\partial\Delta_1$	$\{\{\mathcal{K}\}\}$	1
$\partial\Delta_2$	$\mathcal{P}_{\geq 2}([\mathcal{K}])$	4
$E_4 = \mathcal{K}_1 * \mathcal{K}_2, \mathcal{K}_i = \partial\Delta_1$	$\{\{\mathcal{K}_1\}, \{K_2\}\}$	2
$\partial\Delta_3$	$\mathcal{P}_{\geq 2}([\mathcal{K}])$	11
$\Sigma E_3 = \partial\Delta_1 * \partial\Delta_2$	$\mathcal{B}(\partial\Delta_1) \cup \mathcal{B}(\partial\Delta_2)$	5
$\Sigma E_4 = \partial\Delta_1 * E_4$	$\mathcal{B}(\partial\Delta_1) \cup \mathcal{B}(E_4)$	3
$\Sigma E_n = \partial\Delta_1 * E_n, n \geq 5$	$\{\{\partial\Delta_1\}\}$	1
$\partial C(n,3)$, $n \geq 6$	$\{\{\partial\Delta_1\}\}$	1

Table 5.2: The non-empty $\mathcal{B}(\mathcal{K})$, reproduced from [AC10].

We want to find all degree zero pieces of $T^1_{A_{\mathcal{S}}}$. We do this systematically by considering each of the illustrations in Figure 5.1. There is one figure for each isomorphism class of a link at a vertex, so we need only consider each figure, keeping track of which combinations we count.

Note that there cannot be any contributions in degree $\mathbf{a}-\mathbf{b}$ with $|a| = 3$, because then $\text{link}(a,\mathcal{S}) = \partial\Delta_1$, a simplicial complex having two vertices, and since we must have $b \in \{0,1\}^n$ and $|b| = 3$, this makes no contributions possible.

We first look at the vertex $a = \{x_7\}$ in order to find contributions with $b \subseteq [\text{link}(a,\mathcal{S})]$. See Figure 5.1a). Then Table 5.2 shows that $\mathcal{B}(\text{link}(x_7,\mathcal{S})) = \{\{\partial\Delta_1\}\}$, where $\partial\Delta_1 = \{x_{14}, x_{16}\}$, so that the degree zero element we have is in degree $\frac{x_7^2}{x_{14}x_{16}}$. Since there are two elements in the orbit of $\{x_7\}$, we have 2 elements of this type.

Still considering $\text{link}(\{x_7\},\mathcal{S})$, we consider a with $|a| = 2$, that is, a with the first element x_7 and with the second lying in $\text{link}(\{x_7\},\mathcal{S})$. We see that choosing either $a = \{x_7, x_{16}\}$ or $a = \{x_7, x_{14}\}$ gives $\mathcal{B}(\text{link}(a,\mathcal{S})) = \varnothing$ (because then $\text{link}(a,\mathcal{S})$ is a hexagon, which is not listed in Table 5.2). There are six other possibilities for a second element, and all of these lie on the suspended circle. These come in two types: the ones lying in the size 8 orbit and the ones lying in the size 4 orbit. No matter which element on the suspended circle we choose, we have that $\text{link}(a,\mathcal{S})$ is a quadrilateral. From the table, we see that quadrilaterals contribute with 2 each. In total the a's with $|a| = 2$ and with $\text{link}(a,\mathcal{S})$ a quadrilateral contribute with $6 \cdot 2 \cdot 2 = 24$ elements (number of elements on the circle multiplied with the number of contributions multiplied with the size of the orbit). These are elements of degree of the form $\frac{x_7x_i}{x_jx_k}$ (where x_j and x_k are opposite corners on the quadrilateral $\text{link}(\mathcal{K},\{x_7,x_i\})$).

Now we look at Figure 5.1b. This link, $\text{link}(\{x_8\},\mathcal{S})$ is the suspension of a pentagon. Choosing $a = \{x_8\}$, one finds one element of total degree zero in degree $\frac{x_8^2}{x_2x_6}$. There are 8 isomorphism classes, so we have 8 elements of this type.

Now consider a with $|a| = 2$ and $a \subseteq [\text{link}(\{x_8\},\mathcal{S})]$. We have three possible types of such an a. There are two elements on the suspended circle in the same orbit as x_8, so we get $2 \cdot 2 \cdot 8/2 = 16$ elements of degree, for example, $\frac{x_8x_5}{x_{14}x_{17}}$ and $\frac{x_8x_5}{x_6x_2}$. There are two elements in the same orbit as x_{14}, so we get $2 \cdot 2 \cdot 8 = 32$ elements of degree, for example, $\frac{x_8x_{14}}{x_2x_6}$ and $\frac{x_8x_{14}}{x_5x_7}$. Finally, there is one element in the orbit $\{x_7, x_{11}\}$, but we have already counted this.

Now consider $\text{link}(\{x_{16}\},\mathcal{S})$, seen in Figure 5.1c. Here there are no contributions with $|a| = 1$, so we must look for contributions with $|a| = 2$. Thus all contributions must come from vertices in $\text{link}(\{x_{16}\},\mathcal{S})$ with valency 4. There are two of these, but have already counted them.

All in all, we see that $T^1_{A_{\mathcal{S}},0}$ has a basis consisting of $2+24+8+16+32 = 82$ elements. We sum this up in a proposition:

Proposition 5.2.1. *The space of first-order deformations of* $\mathbb{P}(\mathcal{S})$, $T^1_{A_\mathcal{S},0}$, *is* 82-*dimensional.*

We can describe the basis explicitly: if $\phi \in T_{A_\mathcal{S},\mathbf{c}} \neq 0$ then a generator $x_p \in I_\mathcal{S}$ is mapped to $\phi(x_p) = x^\mathbf{a} x_{p\setminus b}$ if $b \subseteq p$, and 0 otherwise. Since the ideal $I_\mathcal{S}$ is generated by quadratic monomials and all $|b| = 2$, this means the first-order perturbed ideal is generated by $x_\mathbf{b} + \epsilon x_\mathbf{a}$ plus possibly non-perturbed generators.

We are also interested in lower degree contributions. The only contributions with in degree -1 arise when a is a vertex with link$(a, \mathcal{S})$ a n-gon, and they are of the form $\frac{x_i}{x_j x_k}$, and there are ten of them.

From the definition of $\mathcal{B}(\mathcal{S})$, we see that there are no contributions with $a = \varnothing$ and $|b| = 2$. There can't be any contributions with $|a| > 2$.

Summed up:

Lemma 5.2.2. *The degree* -1 *piece of* $T^1_{A_\mathcal{S}}$ *is* 10-*dimensional. All pieces of lower degree vanish.*

5.2.2 The obstruction module, $T^2_{A_\mathcal{S}}$

We want to compute $T^2_{A_\mathcal{S}}$ in degree zero and lower.

Recall that a simplicial complex is *flag* if it has no empty simplices. It is a easy result that a simplicial complex is flag if and only if its Stanley-Reisner ideal is generated by quadratic monomials. This is the case for our simplex, $\mathcal{S}$. In this case, we have the following result from [CI11]:

Proposition 5.2.3. *If* $\mathcal{K}$ *is a flag complex and* $b \in \mathcal{K}$ *and* $|b| \geq 2$, *then* $T^i_{\varnothing - b}(\mathcal{K}) = 0$ *for* $i = 1, 2$.

This means that we don't need to check $b \in \mathcal{S}$, but we still want $\partial b \subset \mathcal{S}$. We first find the degree zero contributions. There are several cases to consider:

- $|a| = |b| = 1$: In this case, b is a vertex, and we have from Proposition 3.4.5 that $T^2_{\mathbf{a}-\mathbf{b}} = \tilde{H}_1(\text{link}(a, \mathcal{S}), k)$. But link$(a, \mathcal{S})$ is always a 2-sphere, and in this case $\tilde{H}_1$ vanishes.
- $|a| = |b| = 2$: We can assume that $b \notin \mathcal{S}$. In this case, the theorem tells us that $T^2_{a-b} = \tilde{H}_{-1}(L_b, k)$, where by convention $\tilde{H}_{-1}(\varnothing, k) = k$. Thus to have a contribution we must have $L_b = \varnothing$. Every link with $|a| = 2$ is a 1-sphere, i.e. an n-gon, and the only n-gons possibly with $L_b = \varnothing$ are those with $n \geq 6$. This is also the the largest n-gon that

occurs in our $\mathcal{K}$, and the only n-gons that occur as links of 2-faces are 5-gons and 6-gons. From the figures, wee see that there are two possible choices in the orbit of $\{x_7\}$, namely $\{x_7, x_{16}\}$ and $\{x_7, x_{14}\}$. The orbit has size two, so there are four contributions of this form. In Figure 5.1c, there are six vertices with valency four, two of which we already have counted. They are $\{x_{16}, x_{10}\}$ and $\{x_{16}, x_6\}$. Applying automorphisms, we see that these two elements constitute an orbit of size four, so that in total there are $4 + 4 = 8$ two-faces with $L_b = \varnothing$. Each 6-gon contributes 3, so we have a total contribution of $3 \cdot 8 = 24$.

- There is also the possibility of $|a| = 1$, but raised to an exponent of 2, so that we must have $|b| = 2$. Then the theorem tells us that $T^2_{a-b} = \tilde{H}_0(L_b, k)$. This means that we must look for links of vertices with L_b having two or more path components. In the link at $\{x_7\}$, we see three such candidates, namely opposite vertices on the circle. This gives 6 contributions. On the figure of the link at $\{x_{16}\}$, we find after close inspection two possibilities: $\{x_6, x_{10}\}$ and $\{x_7, x_{11}\}$. Looking at the orbit of this combination, we see that this gives $2 \cdot 4 = 8$ contributions. In total there are 14 contributions of this type.

One checks that the above possibilities are all that can occur, so one concludes that $T^2_{A_\mathcal{S},0}$ has dimension $24 + 14 = 38$.

To find the contributions in degree -1, one notes that the only possibility is when $|a| = 1$ and $|b| = 2$. But there are fourteen of these, as we counted above.

In degree -2 there are 3 contributions, and they all occur when $a = \varnothing$ and $|b| = 2$. The two first contributions are in degree $\frac{1}{x_6 x_{10}}$ and degree $\frac{1}{x_{14} x_{16}}$, respectively. In each of these cases, L_b looks topologically like $D^2 \vee S^1$, so we have $H^1(L_b, k) = k$. The other contribution occur in degree $\frac{1}{x_7 x_{11}}$, and in this case L_b is topologically a circle, which has $H^1(L_b, k) = k$.

We sum this up in a proposition:

Proposition 5.2.4. *The module $T^2_{A_\mathcal{S}}$ has dimension $3, 14$ and 38 in degree $-2, -1$ and 0, respectively.*

5.3 Construction of the EHG family

We know from the previous chapter that there is a degeneration of the Grassmannian to the Stanley-Reisner ring $P := A_{\mathcal{S} * \Delta^5} = A_\mathcal{S} \otimes_k k[y_1, \cdots, y_6]$. By base extension, we have that $T^1_P = T^1_{A_\mathcal{S}} \otimes_k k[y_1, \cdots, y_6]$, so that

$$\dim_k T^1_{P,0} = 82 + 6 \cdot 10 = 142,$$

since $T^1_{A_\mathcal{S},-1}$ has dimension 10, as calculated above.

Proposition 5.3.1. *The space of first-order deformations, $T^1_{P,0}$, of $\mathbb{P}(\mathcal{S} * \Delta^5)$ is 142-dimensional.*

The obstruction module $T^2_{A_\mathcal{S}}$ was calculated in the previous section in degree zero and below. We know from the deformation theoretic chapter that all obstructions to liftings lie in $T^2_{P,0}$, which by the previous section has dimension $38 + 6 \cdot 14 + 21 \cdot 3 = 185$. The family we construct has obstructions, as we will see below.

It was not computationally feasible to compute a versal deformation, i.e. a deformation using all 142 deformation parameters. What we did, was to choose among the first order deformations some "suitable" deformation parameters. First, we looked for deformation parameters coming from $A_\mathcal{S}$ creating the Hibi relations – there were only 16 such element in $T^1_{A_\mathcal{S},0}$. The group $\mathcal{G} = \mathbb{Z}/2 \times \mathbb{Z}/2$ acts on $T^1_{P,0}$, decomposing it into 21 orbits[1]. The first order deformations creating the Hibi relations constitute four such orbits – by looking for missed monomials we chose two additional orbits - one of size two, containing one element in each of the degrees $\frac{x_{11}x_{20}}{x_6x_{10}}$ and $\frac{x_1x_7}{x_{14}x_{16}}$, and one of size four.

Using the Macaulay2 package `VersalDef` [Ilt11], we used these first-order deformations to lift to higher order. The lifting stops, so we end up with a flat family $\mathcal{X} \to \tilde{\mathcal{T}}$, where the base space $\tilde{\mathcal{T}}$ is a variety defined by binomials. It was computed to be reducible as the union of 13 irreducible toric varieties. The largest irreducible component has dimension 14 and contains the Grassmannian $\mathbb{G}(3,6)$ as a fiber. It has, of course, the Stanley-Reisner scheme $\mathbb{P}(\mathcal{S} * \Delta_5)$ as its special fiber. A degeneration to the Hibi ring is obtained by setting the deformation parameters in two of the six chosen orbits in $T^1_{P,0}$ to zero. Call the component of dimension 14 $\mathcal{T}$. Equations for $\tilde{\mathcal{T}}$ and $\mathcal{T}$ can be found in Appendix C.

We sum this up in a theorem:

Theorem 5.3.2 (The EHG-family)**.** *There exists a family of deformations $\mathcal{X} \to \mathcal{T}$ of the Stanley-Reisner scheme $\mathbb{P}(\mathcal{K} * \Delta_5)$ having the Grassmannian*

[1] Recall that the action was induced by horizontal and vertical mirroring of the lattice $\mathcal{L}_{3,6}$.

$\mathbb{G}(3,6)$ as fiber and as an intermediate fiber the Hibi ring. The base space $\mathcal{T}$ is a 14-dimensional irreducible toric variety.

$$\begin{array}{ccccc} \mathcal{X} & \longleftarrow & \mathcal{X} \times_{\mathcal{T}'} \mathcal{T} & \longleftarrow & \mathbb{P}(\mathcal{K} * \Delta_5) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{T} & \longleftarrow & \mathcal{T}' & \longleftarrow & \operatorname{Spec} k \end{array}$$

Here $\mathcal{T}'$ denotes the restriction of the base space $\mathcal{T}$ such that the generic fiber is the Hibi variety.

We are only interested in the component of the base space that has the Grassmannian as a fiber. It is a toric variety of dimension 14, so that it can be represented by some semigroup ring $k[\mathcal{M}]$, generated by the columns of a matrix $\mathcal{M}$. The ring is not changed by column operations of the matrix $\mathcal{M}$, and we discovered that the column space of the matrix could also be represented by a matrix $\mathcal{M}'$, having the form:

$$\mathcal{M}' = \begin{pmatrix}
1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
1&0&0&0&-1&0&0&0&0&0&1&0&1&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0\\
0&-1&0&0&0&0&0&-1&0&1&0&0&0&0&0&1&0&0&1&0&0&0\\
1&0&0&0&0&0&1&0&0&0&1&0&0&0&0&1&0&0&1&0&0&0
\end{pmatrix}$$

This means that if we identify the parameters in the same orbit, the obstruction ideal vanishes. In other words, the invariant subspace $\mathcal{T}^{\mathcal{G}}$ of $\mathcal{T}$ is affine, isomorphic to $\mathbb{A}^6$. This gives a smooth 6-dimensional family with the Grassmannian as the general fiber.

The special form of $\mathcal{M}'$ implies that the cone associated to $\mathcal{T}$ is the *Cayley cone* over $\Delta_1, \Delta_2, \cdots, \Delta_6$, where the first five Δ_i are 3-simplexes, and Δ_6 is an interval.

From now on, we name the parameters s_i, for $i = 1, \cdots, 6$. We obtain the Grassmannian by setting each $s_i = -1$, and be obtain the Hibi ring by

$s_1 = s_2 = s_4 = s_5 = -1$ and $s_3 = s_6 = 0$. The equations for the invariant family are included in Appendix B.

We would like to analyze the fibers of $\mathcal{X} \to \mathbb{A}^6$. Because of the large number of variables, this was not possible for general $s_i\,(i = 1, \cdots, 6)$. It was possible however over the subspaces where, say, five of the six s_i's were set to zero. Computations here showed that these fibers were invariant up to isomorphism under scaling the deformation parameter. We suspect that this holds generelly: namely, that the fiber over $(a_1, \cdots, a_6) \in \mathbb{A}^6$ is isomorphic to the fiber over $(c_1 a_1, \cdots, c_6 a_6)$ for non-zero c_i. This would be true for example if there were some torus action on $\mathbb{A}^6$ that extends to a torus action on $\mathcal{X}$. We have not however been able to prove the existence of such an action.

Conjecture 5.3.3. *There are only finitely many isomorphism classes of fibers in the family $\mathcal{X} \to \mathbb{A}^6$.*

5.3.1 The fibers of the invariant family

In the rest of the chapter we will analyze the fibers of $\mathcal{X} \to \mathcal{T}^{\mathcal{G}} = \mathbb{A}^6$ when each $s_i \in \{0, -1\}$. When referring to specific fibers, we will use the notation X_{123}, for the fiber where $s_1 = s_2 = s_3 = -1$, and $s_4 = s_5 = s_6 = 0$, et cetera.

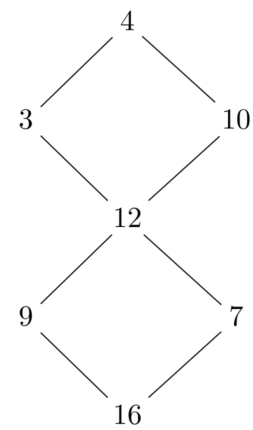
Most of the fibers are reducible and most of them share isomorphic components. Many of the components of the fibers admit nice descriptions in terms of sublattices of $\mathcal{L}_{3,6}$ or in terms of *punctured matrices*. A punctured matrix M is a matrix with entries in not all positions (see the next few pages for examples). The variety associated to M is the zero set of a maximal set of 2×2-minors. That is, the zero set of all $\{x_{ij}x_{kl} - x_{il}x_{kj}\}$ such that all $M(i,j), M(k,l), M(i,l), M(k,j)$ are defined. They can be seen to correspond to distributive lattices (in such a way that the Hibi ring associated to the distributive lattice is equal to the coordinate ring of the zero set of the 2×2-minors of the punctured matrix).

Example 5.3.4. The punctured matrix on the left in Figure 5.2 corresponds to the the distributive lattice on the right. One sees that the correspondence is far from one-to-one. $\diamondsuit$

We first describe what type of of components of fibers occur. Recall that to obtain the Hibi ring, one sets the deformation parameters s_1, s_2, s_4, s_5 all equal to -1. If $\mathcal{A} = \{a_1, \cdots, a_n\}$ is the vector configuration of the order

$$\begin{vmatrix} - & 3 & 4 \\ 9 & 12 & 10 \\ 16 & 7 & - \end{vmatrix}$$

(a) A punctured matrix.



(b) A distributive lattice.

Figure 5.2: Correspondence between punctured matrices and distributive lattices.

polytope (corresponding to the Hibi ring), then the ring $k[x_1, \cdots, x_n]$ is $\mathcal{A}$-graded, by setting each $\deg x_i = a_i$. So $I_{\mathcal{A}}$ is an $\mathcal{A}$-graded ideal, and it follows that the initial ideal $in_<(I_{\mathcal{A}})$ is $\mathcal{A}$-graded as well.

One can check that the deformation parameters all have weight zero in the $\mathcal{A}$-grading. It follows that the ideal obtained by using these deformation parameters is $\mathcal{A}$-graded. We are therefore in position to apply a theorem of Sturmfels:

Theorem 5.3.5 (Sturmfels)**.** *If I is any $\mathcal{A}$-graded ideal, then there is a polyhedral subdivision Δ of $\mathcal{A}$ such that*

$$\sqrt{I} = \bigcap_{\sigma \in \Delta} J_\sigma,$$

where J_σ is a prime ideal torus isomorphic to I_σ. (here I_σ denotes the ideal $I_\sigma + \langle x_i | a_i \notin \sigma \rangle$)

Proof. See [Stu96]. □

In our case, all fibers are radical, so we can forget about the square root sign. What this mean is the following: All fibers obtained using the deformation parameters $\{s_1, s_2, s_4, s_5\}$ correspond to polyhedral subdivisions of $\mathcal{A}$. A minimal polyhedral subdivision will be just the cone of $\mathcal{A}$, and a maximal polyhedral subdivision corresponds to a triangulation of the associated simplicial complex.

This is good for visualization purposes: What happens when we deform the Stanley-Reisner ring is that linear spaces are "glued" together successively until we arrive at the Hibi ring. In cone-language: (cones of) triangles in the triangulation merge together until there is only one cone left.

Note that this confirms our Conjecture 5.3.3 over the locus where $s_3 = s_6 = 0$. We computed all the fibers when $s_i \in \{0, -1\}$. We state this as a theorem:

Theorem 5.3.6. *The fibers in the family $\mathcal{X} \to \mathcal{T}^{\mathcal{G}}|_{s_3=s_6=0}$ are the schemes listed in Table 5.3.*

Assuming Conjecture 5.3.3, we have:

Theorem 5.3.7. *The fibers in the family $\mathcal{X} \to \mathcal{T}^{\mathcal{G}}$ are the schemes listed in Tables 5.3 and 5.4. There are some irreducible fibers that is not isomorphic to $H_{\mathcal{L}_{3,6}}$, and we describe those in the next section.*

After the tables we explain the notation.

Fiber	Isomorphism class	# of this type
X_1	$\mathbb{P}^9$	26
	$C_7(\mathbb{P}^1 \times \mathbb{P}^1)$	8
X_2	$\mathbb{P}^9$	26
	$C_7(\mathbb{P}^1 \times \mathbb{P}^1)$	8
X_4	$\mathbb{P}^9$	10
	$C_7(\mathbb{P}^1 \times \mathbb{P}^1)$	2
	$C_3(S_1)$	12
X_5	$\mathbb{P}^9$	10
	$C_7(\mathbb{P}^1 \times \mathbb{P}^1)$	2
	$C_3(S_1)$	12
X_{12}	$\mathbb{P}^9$	14
	$C_7(\mathbb{P}^1 \times \mathbb{P}^1)$	8
	$C_6(\mathbb{P}^1 \times \mathbb{P}^2)$	4
X_{14}	$\mathbb{P}^9$	10
	$C_7(\mathbb{P}^1 \times \mathbb{P}^1)$	4
	$C_6(S_1)$	2
	$C_3(S_2)$	4
X_{24}	$\mathbb{P}^9$	2
	$C_7(\mathbb{P}^1 \times \mathbb{P}^1)$	4
	$C_6(S_1)$	2
	$C_6(\mathbb{P}^1 \times \mathbb{P}^2)$	8

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Table 5.3 – *Continued from previous page*

Fiber	Isomorphism class	# of this type
X_{25}	$\mathbb{P}^9$	2
	$C_7(\mathbb{P}^1 \times \mathbb{P}^1)$	4
	$C_6(S_1)$	2
	$C_6(\mathbb{P}^1 \times \mathbb{P}^2)$	8
X_{15}	$C_7(\mathbb{P}^1 \times \mathbb{P}^1)$	2
	$C_4(P_1)$	2
	$C_5(P_2)$	4
X_{45}	$C_2(P_3)$	2
	$C_3(P_4)$	2
X_{124}	$\mathbb{P}^9$	2
	$C_7(\mathbb{P}^1 \times \mathbb{P}^1)$	4
	$C_5(P_2)$	4
X_{125}	$C_6(T_1)$	2
	$C_6(T_2)$	2
X_{145}	$C_6(T_3)$	2
X_{245}	$C_7(T_4)$	2
	$C_8(T_5)$	2
X_{1245}	Hibi.	1

Table 5.3: Degenerations of the Hibi ring.

Fiber	Isomorphism class	# of this type
X_3	$\mathbb{P}^9$	26
	$C_7(\mathbb{P}^1 \times \mathbb{P}^1)$	8
X_6	$\mathbb{P}^9$	18
X_{16}	$\mathbb{P}^9$	2
	$C_7(\mathbb{P}^1 \times \mathbb{P}^1)$	4
	$C_6(\mathbb{P}^1 \times \mathbb{P}^2)$	8
	$C_6(S_1)$	2
X_{13}	$\mathbb{P}^9$	14
	$C_7(\mathbb{P}^1 \times \mathbb{P}^1)$	8
	$C_6(\mathbb{P}^1 \times \mathbb{P}^2)$	4
X_{23}	$\mathbb{P}^9$	14
	$C_7(\mathbb{P}^1 \times \mathbb{P}^1)$	8
	$C_6(\mathbb{P}^1 \times \mathbb{P}^2)$	4

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Table 5.4 – *Continued from previous page*

Fiber	Isomorphism class	# of this type
X_{26}	$\mathbb{P}^9$	18
	$C_7(\mathbb{P}^1 \times \mathbb{P}^1)$	4
	$C_3(S_2)$	4
X_{34}	$C_4(P_1)$	2
	$C_5(P_2)$	4
	$C_6(\mathbb{P}^1 \times \mathbb{P}^2)$	2
X_{35}	$\mathbb{P}^9$	10
	$C_7(\mathbb{P}^1 \times \mathbb{P}^1)$	4
	$C_6(S_1)$	2
	$C_3(S_2)$	4
X_{36}	$\mathbb{P}^9$	2
	$C_7(\mathbb{P}^1 \times \mathbb{P}^1)$	4
	$C_6(S_1)$	2
	$C_6(\mathbb{P}^1 \times \mathbb{P}^2)$	8
X_{46}	$\mathbb{P}^9$	2
	$C_7(\mathbb{P}^1 \times \mathbb{P}^1)$	8
	$C_6(S_1)$	2
X_{56}	$\mathbb{P}^9$	2
	$C_7(\mathbb{P}^1 \times \mathbb{P}^1)$	8
	$C_6(S_1)$	6
X_{123}	$C_6(S_4)$	2
	$C_7(\mathbb{P}^1 \times \mathbb{P}^1)$	12
	$\mathbb{P}^9$	6
X_{126}	$C_5(P_2)$	4
	$C_6(\mathbb{P}^1 \times \mathbb{P}^2)$	4
	$\mathbb{P}^9$	2
X_{134}	$C_4(P_1)$	2
	$C_6(\mathbb{P}^1 \times \mathbb{P}^2)$	2
	$C_4(J(\cap_2^2 \mathbb{G}(2,4), \mathbb{P}^1 \times \mathbb{P}^2))$	2
X_{135}	$C_4(P_1)$	2
	$C_6(\mathbb{P}^1 \times \mathbb{P}^2)$	2
	$C_4(J(\cap_2^2 \mathbb{G}(2,4), \mathbb{P}^1 \times \mathbb{P}^2))$	4
X_{136}	$C_4(P_5)$	2
	$C_4(\mathbb{P}^1 \times \mathbb{P}^4)$	2
X_{146}	$\mathbb{P}^9$	2
	$C_7(\mathbb{P}^1 \times \mathbb{P}^1)$	4

Continued on next page

Table 5.4 – *Continued from previous page*

Fiber	Isomorphism class	# of this type
	$C_2(\cap_2^2\mathbb{G}(2,4))$	2
	$C_4(T_6)$	2
X_{156}	$C_3(T_7)$	2
X_{234}	$C_4(P_5)$	2
	$C_4(P_6)$	2
X_{235}	$\mathbb{P}^9$	2
	$C_7(\mathbb{P}^1\times\mathbb{P}^1)$	2
	$C_6(\mathbb{P}^1\times\mathbb{P}^2)$	4
	$C_5(P_2)$	4
X_{236}	$C_7(S_1)$	2
	$C_6(\mathbb{P}^1\times\mathbb{P}^2)$	4
	$\mathbb{P}^9$	2
X_{246}	$C_6(S_1)$	2
	$C_7(\mathbb{P}^1\times\mathbb{P}^1)$	4
	$C_3(T_7)$	4
	$\mathbb{P}^9$	2
X_{256}	$C_6(S_1)$	2
	$C_7(\mathbb{P}^1\times\mathbb{P}^1)$	4
	$C_3(T_7)$	4
	$\mathbb{P}^9$	2
X_{345}	$C_3(T_7)$	2
X_{346}	$C_3(T_7)$	2
X_{356}	$C_4(T_6)$	4
	$C_7(\mathbb{P}^1\times\mathbb{P}^1)$	6
	$\mathbb{P}^9$	2
X_{456}	$C_2(P_3)$	4
	$C_8(T_8)$	2
X_{1234}	P_5	2
	T_9	2
X_{1235}	$C_2(P_3)$	2
	$C_1((\mathbb{P}^2\times\mathbb{P}^2)\cap_1\mathbb{G}(2,4)$ (one variable in common)	2
X_{1236}	$C_4(P_5)$	2
	$C_4(T_8)$	2
X_{1246}	$\mathbb{P}^9$	2
	$C_4(P_1)$	4

Continued on next page

Table 5.4 – *Continued from previous page*

Fiber	Isomorphism class	# of this type
	$C_2(\cap_2^2 \mathbb{G}(2,4))$	2
X_{1256}	Hibi.	1
X_{1345}	T_{10} See Appendix C.	2
X_{1346}	T_{10}. See Appendix C.	1
X_{1356}	T_{10}. See Appendix C.	2
X_{1456}	$P_1 \bigcap J^2(\cap_2^2 \mathbb{G}(2,4))$	2
X_{2345}	Hibi.	1
X_{2346}	Hibi.	1
X_{2356}	$\mathbb{P}^9$ $C_4(P_1)$ $C_2(\cap_2^2 \mathbb{G}(2,4))$	2 4 2
X_{2456}	$C_1(T_{11})$ $C_2(P_3)$	2 2
X_{3456}	$P_1 \cap^4 \mathbb{G}(2,4)$	2
X_{13456}	$\mathbb{P}^9$ $C_4(P_1)$ $C_2(\cap_2^2 \mathbb{G}(2,4))$	2 4 2

Table 5.4: Isomorphism classes of degenerations of the Grassmannian $\mathbb{G}(3,6)$.

The fibers using the Hibi parameters in Table 5.3. The other components are described in Table 5.4. The next few pages are explanations on the notation in the tables.

Explanation: $\mathbb{P}^9$

The ideal of a $\mathbb{P}^9$ is given by ten variables, with ten variables free. They come from faces of the complex $\mathcal{S} * \Delta^5$.

Explanation: $C_i(X)$

By $C_i(X)$ we mean the *cone* over a variety. In equations this means that there are i free variables. So, for example, consider $C_7(\mathbb{P}^1 \times \mathbb{P}^1)$. There are a total of 20 variables, 4 of them are used in the Segre equation $(x_i x_j - x_k x_l)$, there are 7 free variables, so the ideal is generated by 9 variables plus the Segre equation.

Explanation: S_1

S_1 is the join of two disjoint copies of $\mathbb{P}^1 \times \mathbb{P}^1$. Recall that the *join* of two varieties is the union of lines connecting points from each variety. In equations it is given by the zero locus of two 2×2-determinants with disjoint variables.

Explanation: S_2

S_2 can be described as the zero locus of the 2×2-minors of the following punctured matrix:

$$\begin{vmatrix} - & x_3 & x_4 \\ x_9 & x_{12} & x_{10} \\ x_{16} & x_7 & - \end{vmatrix}$$

Notice that S_2 is of Hibi type.

Explanation: P_1

The two P_1's are the zero locus of the ideal generated by the 2×2-minors of the punctured matrix below.

$$\begin{vmatrix} - & x_{10} & x_2 \\ x_{18} & x_{11} & x_{17} \\ x_7 & x_{13} & x_8 \\ x_{14} & - & x_5 \end{vmatrix}$$

Explanation: P_2

The P_2's are zero loci of 2×2 minors of punctured matrices of the form:

$$\begin{vmatrix} x_7 & x_8 & x_{13} \\ x_{18} & x_{17} & x_{11} \\ - & x_2 & x_{10} \end{vmatrix}$$

Note that this ideal is also of Hibi type.

Explanation: P_3

P_3 is the zero locus of the 2×2 minors of the following matrices:

$$\begin{vmatrix} x_{19} & x_{14} & x_5 \\ x_{16} & x_{18} & x_{17} \end{vmatrix} \text{ and } \begin{vmatrix} x_8 & x_{13} & x_6 \\ x_2 & x_{10} & x_4 \end{vmatrix}.$$

We see that P_3 is the join of two disjoint copies of $\mathbb{P}^1 \times \mathbb{P}^2$.

Explanation: P_4

P_4 is the zero locus of the 2×2 minors of the following matrices:

$$\begin{vmatrix} x_{19} & x_{14} \\ x_{16} & x_{18} \end{vmatrix} \text{ and } \begin{vmatrix} x_8 & x_{13} & x_6 \\ x_2 & x_{10} & x_4 \end{vmatrix}$$

Note that P_4 is the join of $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \mathbb{P}^2$. It is also of Hibi type.

Explanation: T_1

The ideal T_1 is the lattice ideal associated to the lattice in Figure 5.4. Notice that it is obtained from $\mathcal{L}_{3,6}$ (Figure 5.3) by removing the vertices $9, 16, 3$ and 6. See Figure 5.4.

Explanation: T_2

T_2 is obtained by removing the vertices 10 and 2 instead of 3 and 6 from $\mathcal{L}_{3,6}$.

Explanation: T_3

The ideal of T_3 is the lattice ideal obtained from $\mathcal{L}_{3,6}$ by removing the right-most vertices.

Explanation: T_4

The ideal of T_4 is the lattice ideal obtained from $\mathcal{L}_{3,6}$ by removing the vertices $2, 3, 8, 12, 7$. See figure 5.5. Note that this results in the join of $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2 \times \mathbb{P}^2$.

Explanation: T_5

The ideal of T_5 is obtained by removing the vertices $7, 11, 5, 17, 8, 2$ from $\mathcal{L}_{3,6}$. This results in the join of two disjoint copies of $\mathbb{P}^1 \times \mathbb{P}^2$. See Figure 5.6.

Explanation: T_6

The ideal of T_6 is the lattice ideal of the lattice in Figure 5.7.

Explanation: T_7

The ideal of T_7 is a lattice ideal, as it is the zero locus of the following punctured matrix:

$$\begin{vmatrix} x_1 & x_{16} & x_{17} & - & - \\ x_{14} & x_7 & x_2 & x_4 & - \\ - & x_{19} & x_5 & x_{11} & x_{10} \\ - & - & x_8 & x_6 & x_{20} \end{vmatrix}.$$

Explanation: T_8

T_8 is the join of $\mathbb{P}^1 \times \mathbb{P}^2$ and a copy of $\mathbb{G}(2,4)$. Explicitly, its equations are

$$\text{rank} \begin{vmatrix} x_{15} & x_{18} & x_{14} \\ x_9 & x_{16} & x_{19} \end{vmatrix} \leq 1 \text{ and } x_6 x_{10} - x_{13} x_4 + x_{11} x_{20} = 0.$$

Explanation: T_9

T_9 is projective variety given by the maximal minors of the matrix below:

$$\begin{pmatrix} x_9 & x_{16} & x_{12} & x_{19} & x_3 & x_7 \\ x_{11} & x_{17} & x_{10} & x_5 & x_5 & x_2 \end{pmatrix}$$

The quadrics:

- $x_4 x_5 - x_{11} x_2 + x_{10} x_{17}$.
- $x_7 x_9 - x_{12} x_{16} - x_3 x_{19}$.
- $x_7 x_{11} - x_{12} x_{17} - x_4 x_{19}$.
- $x_2 x_9 - x_{12} x_{17} - x_4 x_{19}$.

Explanation: T_{10}

I have not been able to find a reasonable pretty description of this component. Therefore I have included its equations in Appendix C. Note that this component occurs in more than one fiber.

Explanation: T_{11}

This can be described as $(\mathbb{P}^2 \times \mathbb{P}^2) \cap \mathbb{G}(2,4)$. In equations, it is given by the 2×2-minors of the matrix below, plus the quadric $x_{10} x_6 - x_4 x_{13} - x_{11} x_{20}$.

$$\begin{pmatrix} x_9 & x_{16} & x_{19} \\ x_{15} & x_{18} & x_{14} \\ x_{11} & x_{17} & x_5 \end{pmatrix}.$$

Explanation: S_3

S_3 is the intersection of three not disjoint copies of $\mathbb{P}^1 \times \mathbb{P}^2$. Explicitly, it is given by the 2×2-minors of the following matrices:

$$\begin{pmatrix} x_{10} & x_7 & x_{12} \\ x_{11} & x_{16} & x_9 \end{pmatrix}, \begin{pmatrix} x_2 & x_7 & x_{17} \\ x_{10} & x_{12} & x_{11} \end{pmatrix}, \text{ and } \begin{pmatrix} x_{16} & x_7 & x_9 \\ x_{17} & x_2 & x_{11} \end{pmatrix}.$$

Explanation: $\cap_2^2 G(2,4)$

This is the intersection of two copies of $G(2,4)$, along two coordinate axes. The equations look like, up to renaming of variables:

$$x_1x_{12} - x_9x_{14} + x_{15}x_{19}$$
$$x_4x_{12} - x_3x_{10} + x_{15}x_{20}$$

Explanation: P_5

The ideal of P_5 is given by the the 2×2-minors of the following two matrices:

$$\begin{pmatrix} x_3 & x_{11} & x_{15} & x_9 & x_4 \\ x_7 & x_5 & x_{14} & x_{19} & x_2 \end{pmatrix} \text{ and } \begin{pmatrix} x_5 & x_{10} & x_2 & x_4 & x_{11} \\ x_{19} & x_{12} & x_7 & x_3 & x_9 \end{pmatrix}.$$

Explanation: P_6

The ideal of P_6 is given by the the 2×2-minors of the following two matrices:

$$\begin{pmatrix} x_4 & x_{11} & x_{17} & x_{10} & x_5 & x_2 \\ x_3 & x_9 & x_{16} & x_{12} & x_{19} & x_7 \end{pmatrix} \text{ and } \begin{pmatrix} x_7 & x_4 & x_3 & x_2 \\ x_{19} & x_{11} & x_9 & x_5 \end{pmatrix}.$$

Explanation: $J^n(X)$ **and** $J(X,Y)$

This is the iterated (disjoint) *join* of the variety X. This means that if X are given by $f_1 = 0, f_2 = 0, \cdots, f_r = 0$, then $J^2(X)$ is given by the same equations, but with different variables. Note that with this notation, S_1 is $J^2(\mathbb{P}^1 \times \mathbb{P}^1)$.

Similarly, $J(X,Y)$ means the disjoint join X and Y.

5.4 The irreducible fibers

If one uses five of the six deformation parameters, there are six possible fibers. Five of them are irreducible, and it turns out that of these five there are two isomorphism classes. These five are X_{12345}, X_{12346}, X_{12356}, X_{12456} and X_{23456}.

We are able to describe their singular locus.

Theorem 5.4.1. *The variety X_{23456}, is irreducible, with singular locus equal to the union of two copies of $C_4(\mathbb{P}^1 \times \mathbb{P}^2)$ and four $C_1(\mathbb{P}^1 \times \mathbb{P}^2)$ (both in their Segre embeddings) .*

Proof. That X_{23456} is irreducible, follows since it degenerates to X_{2345}, which is isomorphic to the Hibi variety, which of course is irreducible.

The rest of the proof will be a description of the strategy used to compute the result.

The variety is embedded in $\mathbb{P}^{19}$, and so has a natural open affine cover, obtained by successively setting each variable equal to 1. Being singular is a local condition, so it is enough to check singularity in each chart.

What we see, is that putting a variable equal to one simplifies the equations significantly, so that we can use the Jacobian criterium in each chart. This gives us local equations for the singular locus. We then take the preimage of this ideal under the localization map and homogenizes by the variable we set to 1 (this corresponds to taking the closure). If we do this in each chart, we will get the whole singular locus.

In some of the charts, the equations didn't simplify enough. For example, in the chart U_{13}, the equations were

$$
\begin{array}{ll}
x_{11}x_{16} - x_9x_{17} & x_{10}x_{16} - x_{12}x_{17} \\
x_5x_{16} - x_{17}x_{19} & x_{11}x_{14} - x_5x_{15} \\
x_9x_{14} - x_{15}x_{19} & x_6x_{14} - x_8x_{15} \\
x_5x_{12} - x_{10}x_{19} & x_9x_{10} - x_{11}x_{12} \\
x_8x_9 - x_6x_{19} & x_5x_9 - x_{11}x_{19} \\
x_5x_6 - x_8x_{11} & \\
x_1x_8x_{12} - x_1x_{19}x_{20} - x_{14}x_{16} + x_{18}x_{19} & \\
x_1x_6x_{12} - x_1x_9x_{20} - x_{15}x_{16} + x_9x_{18} & \\
x_1x_8x_{10} - x_1x_5x_{20} - x_{14}x_{17} + x_5x_{18} & \\
x_1x_6x_{10} - x_1x_{11}x_{20} - x_{15}x_{17} + x_{11}x_{18} &
\end{array}
$$

These were too complicated for `Macaulay2` to compute directly. We first computed the Jacobian matrix (this is not a computation heavy operation).

Since we knew by previous computations that some charts were non-singular, we could substitute these variables by zero. After having deleted some rows and columns with only zeros, we are left with a 13×14-matrix.

To find the singular locus, we must calculate all its 6×6-minors (this is because the ambient space, after having deleted unused variables is 15-dimensional). This computation takes about 5 hours on a computer, and we get a list of 590592 non-zero minors. Amazingly, `Macaulay2` is able to perform a `mingens` operation, so we find that the ideal is minimally generated by 2762 elements. To compute the radical of this ideal, we take the radical of the ideal generated by the first 100 elements, and so on. We finally sum all these radical ideas, take the radical again, and we end up with local equations for the singular locus in the chart U_{13}.

These can be pulled back and homogenized, and they are isomorphic to $C_1(\mathbb{P}^1 \times \mathbb{P}^1)$. It turns out that the two copies of $C_1(\mathbb{P}^1 \times \mathbb{P}^1)$ are isomorphic via the isomorphism of $\mathbb{P}^{19}$ that mirrors the lattice of $\mathbb{G}(3,6)$ vertically. □

Theorem 5.4.2. *The variety X_{12345} has as singular locus two components isomorphic to $C_2((\mathbb{P}^2 \times \mathbb{P}^2))$, thus of dimension $4+2=6$. They intersect along $x_1 = x_{20} = 0$.*

Proof. This is proved in exactly the same way as the previous theorem, except that the computations are easier. One checks on each chart locally, homogenizes, et cetera. The computations are made easier by using the isomorphism of the fibers coming from turning the lattice of $\mathbb{G}(3,6)$ upside down. □

Finally, we have that these two fibers are the only isomorphism classes of irreducible fibers using five deformation parameters:

Theorem 5.4.3. *There are isomorphisms $X_{12345} \simeq X_{12346} \simeq X_{12356}$, and an isomorphism $X_{12456} \simeq X_{23456}$.*

Proof. The isomorphism comes from a permutation of the variables in $\mathbb{P}^{19}$ preserving the ideals. The strategy was to write up the binomial equations in each ideal and guess an isomorphism. □

During the computations we found many isomorphisms $\mathbb{P}^{19} \to \mathbb{P}^{19}$ over order two, inducing isomorphisms on fibers. Many of them came from reflections of the lattice $\mathcal{L}_{3,6}$, and many didn't. This last fact may indicate the existence of an involution $\mathcal{X} \to \mathcal{X}$. I have however not been able to find it.

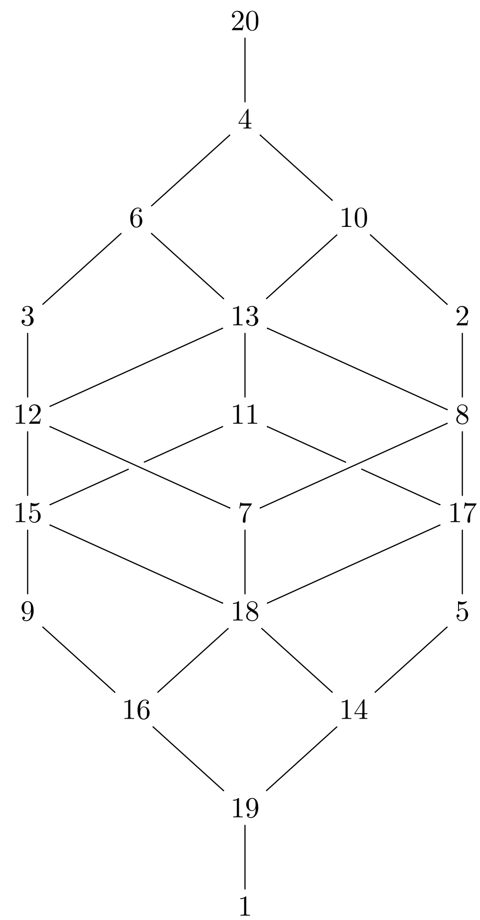

Figure 5.3: The distributive lattice $\mathcal{L}_{3,6}$, renamed.

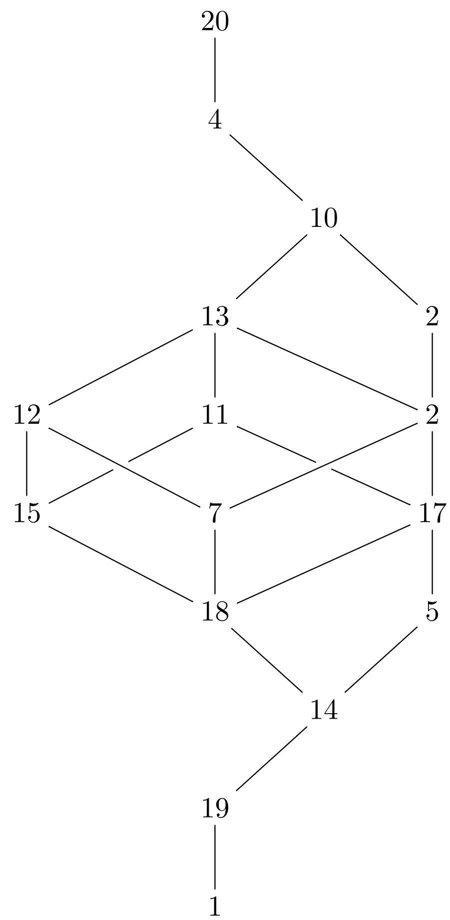

Figure 5.4: The distributive lattice of T_1.

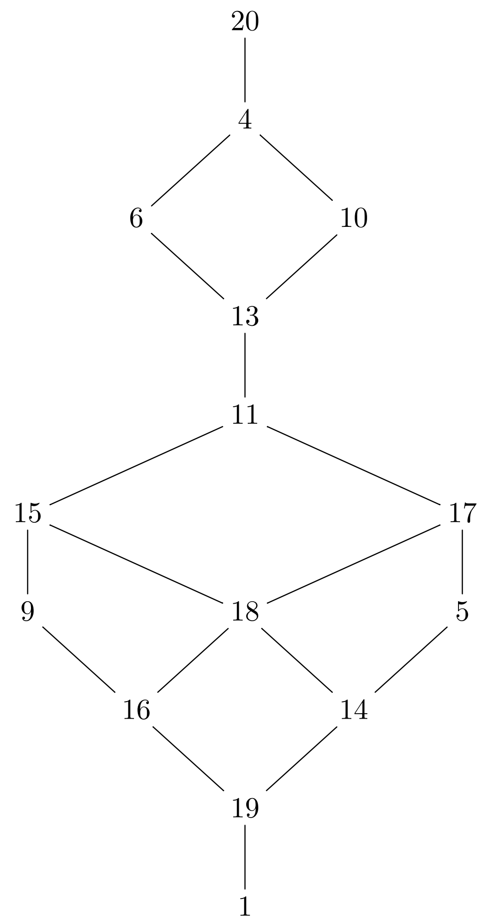

Figure 5.5: The distributive lattice of T_4.

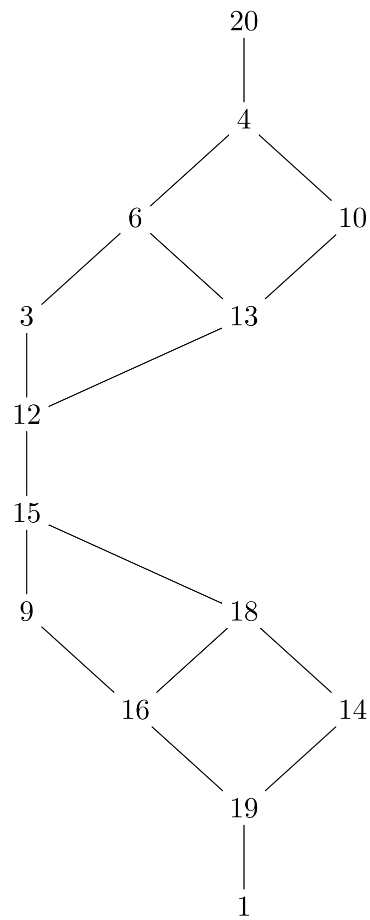

Figure 5.6: The distributive lattice of T_5.

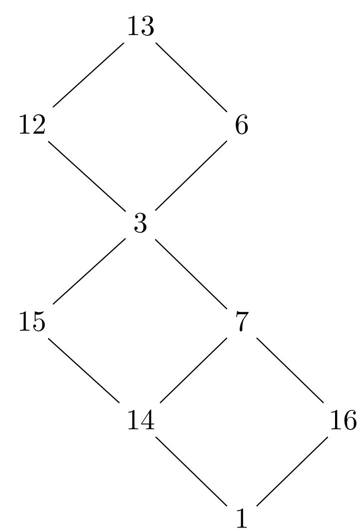

Figure 5.7: The distributive lattice of T_6.

Appendix A

Decomposition techniques

We will briefly describe the techniques used to decompose the various types of ideals encountered during our computations.

A.1 Binomial ideals

If the ideal is binomial (in a general sense: we also call monomial ideals for binomial), there is an extremely efficient algorithm by Eisenbud-Sturmfels, as described in [ES96]. This algorithm is implemented in `Macaulay2`.

A.2 Colon-ideals

If one of the minimal generators of an ideal factors, then the ideal is not prime. A very useful technique relies on the following (easily proven) lemma:

Lemma A.2.1. *If* $(I : f^\infty) = (I : f^k)$*, then* $I = (I : f^\infty) \cap (J, f^k)$.

Proof. See [DE05, Chapter 5]. □

If we know that the ideal I is Cohen-Macaulay, then it is equidimensional. This, combined with good choices of splitting polynomials for I, gives an effective algorithm for decomposing I provied each step succeeds: 1) Look for splitting polynomials. 2) Compute $(I : f^\infty), (I : f^k)$ and (I, f^k). Repeate. The process eventually stops.

A polynomial f satisfying the condition in the lemma is called a *splitting polynomial.* In all the reducible ideals we encountered, it was easy to guess a splitting polynomial because it occured among the generators of the ideal. In general it is a hard problem to find splitting polynomials of reducible ideals.

A.3 Brute force with `Macaulay2`

This is only feasible if the ideal contains lot of monomials or has few generators, in which case the algorithm of `Macaulay2` fast enough.

If one cannot find a splitting polynomial and the ideal has many generators, it is generally not feasible to decompose it using `Macaulay2`. One has to either guess a splitting polynomial, or use other information about the ideal. If for example one knows that the ideal degenerates to a prime ideal, then it follows immedietaly that the ideal was prime, since the number of components is upper semicontinous.

Appendix B

Computer code

A lot of experimentation and exploration has been done with the help of the computer algebra system `Macaulay2` [GS]. Here we include some of the relevant computer code.

B.1 T^1_A and T^2_A

The following code can be used to compute a basis for $T^1(A, A/k)$ in degree n. P is a polynomial ring and I as an ideal such that $A = P/I$.

```
C     = res(I, LengthLimit => 2);
A     = P/I
N     = Hom(image C.dd_1, A)
dI    = sub(transpose jacobian C.dd_1,A)
T1    = N/image dI
B     = matrix basis(n,T1)
T1Mat = (gens N) * B
```

The following code generates a basis for $T^2(A, A/k)$ in degree n.

```
C     = res(I, LengthLimit => 2);
Rel0  = koszul(2, C.dd_1);
A     = P/I;
HomR  = Hom(image(C.dd_2)/image Rel0**A, A);
Triv  = image substitute(transpose C.dd_2, A);
T2    = HomR/Triv;
B     = matrix basis(n, T2);
T2Mat = (gens HomR) * B
```

The output is a matrix. Each column in the matrix represents a homomorphism $R \to A$, where R is the module of relations of I.

Remark. *The package* `VersalDeformation` *written by Nathan Ilten has implemented methods to compute the* T^i *modules in any degree. The notation is* `CT^i(n,F)` *for* $i = 1, 2$ *and* n *an integer.* `F` *is the matrix of generators of the ideal, obtained simply as* `gens I`.

B.2 Finding the flat family

The package `VersalDeformations` [Ilt11] can compute versal deformation spaces, though this is often impossible due to limited computer power. It was however possible in our case, where we didn't use all deformation parameters. Below is the code used to construct the family $\mathcal{T}'$ of Chapter 5.

```
F0 = gens I;
R0 = syz F0;
F1 = firstOrderDefMatrix(F0,Ts | Us, LoL | LoLU)
R1 = (-F1*R0) // F0;
T2 = cotanComplexTwo(0,F0);
C = {sub(T2, ring F0),0};
G = {}
(F,R,G,C)=liftDeformation({F0,F1},{R0,R1},G,C);
(F,R,G,C)=liftDeformation(F,R,G,C);
(F,R,G,C)=liftDeformation(F,R,G,C);
(F,R,G,C)=liftDeformation(F,R,G,C);
(F,R,G,C)=liftDeformation(F,R,G,C);
(F,R,G,C)=liftDeformation(F,R,G,C);
(F,R,G,C)=liftDeformation(F,R,G,C);
```

`F1` is a matrix of first order deformations. The lifting procedure ends. The output `(F,R,G,C)` is the data of a deformation family. The equations for the family are obtained as `sum F`, and the equations of the base space are obtained as `sum G`. The family satisfies

```
transpose ( (sum F)*(sum R))+(sum C)*sum(G)==0
```

B.3 Presentations of toric ideals

A binomial prime ideal expresses the relation between points of a polytope that is the convex hull of the columns of a matrix M. It is well known how to obtain this matrix from the presentation of the ideal, but we include our implementation here for completeness:

```
makeCone = method()
makeCone(Ideal) := (I) -> (
     mGens := mingens I;
     M  := {};
     lll := {};
     for j from 0 to (numColumns(mGens)-1) do (
        lll = exponents (mGens)_j_0;
  if (length lll == 1) then (
       M = M | {lll#0};
       )
  else if (length lll == 2) then (
       M = M | {(lll#0-lll#1)};
       )
        );
     B := transpose matrix M;
     transpose LLL syz  matrix transpose B
     )
```

The result is a `Macaulay2` method that takes a binomial ideal as input and outputs a matrix.

Appendix C

Equations

C.1 The family $\mathcal{X} \to \tilde{\mathcal{T}}$

The family $\mathcal{X} \to \tilde{\mathcal{T}}$ has the set of 35 equations presented in Figure C.2. The parameters can be sorted into 6 orbits. Orbit 1 consists of the parameters $\{s_{63}, s_{66}, s_{68}, s_{70}\}$. Orbit 2 consists of the parameters $\{s_{65}, s_{74}, s_{77}, s_{82}\}$. Orbit 3 consists of the parameters $\{s_{73}, s_{76}, s_{79}, s_{81}\}$. Orbit 4 consists of the parameters $\{u_8, u_{12}, u_{15}, u_{17}\}$. Orbit 5 consists of the parameters $\{u_2, u_3, u_9, u_5\}$. Orbit 6 consists of the parameters $\{x_{11}, x_7\}$.

C.2 The base space $\mathcal{T}$

The equations for $\mathcal{T} \subset \mathbb{A}^{22}$:

$$
\begin{array}{ll}
u_{15}u_9 - u_{17}u_5 & s_{82}u_9 - s_{77}u_5 \\
u_8u_2 - u_{12}u_3 & s_{74}u_2 - s_{65}u_3 \\
s_{77}u_{15} - s_{82}u_{17} & s_{65}u_8 - s_{74}u_{12} \\
s_{82}s_{76} - s_{74}s_{81} & s_{66}s_{76} - s_{68}s_{81} \\
s_{77}s_{73} - s_{65}s_{79} & s_{63}s_{73} - s_{70}s_{79} \\
s_{66}s_{74} - s_{68}s_{82} & s_{63}s_{65} - s_{70}s_{77} \\
s_{66}s_{70}u_3u_9 - s_{63}s_{68}u_2u_5 & s_{76}s_{79}u_{12}u_{15} - s_{73}s_{81}u_8u_{17}
\end{array}
$$

The equations for the 14-dimensional component:

$u_{15}u_9 - u_{17}u_5$	$s_{82}u_9 - s_{77}u_5$	$u_8u_2 - u_{12}u_3$
$s_{74}u_2 - s_{65}u_3$	$s_{77}u_{15} - s_{82}u_{17}$	$s_{65}u_8 - s_{74}u_{12}$
$s_{82}s_{76} - s_{74}s_{81}$	$s_{66}s_{76} - s_{68}s_{81}$	$s_{77}s_{73} - s_{65}s_{79}$
$s_{63}s_{73} - s_{70}s_{79}$	$s_{66}s_{74} - s_{68}s_{82}$	$s_{63}s_{65} - s_{70}s_{77}$
$s_{73}s_{81}u_3u_9 - s_{76}s_{79}u_2u_5$	$s_{70}s_{81}u_3u_9 - s_{63}s_{76}u_2u_5$	$s_{66}s_{73}u_3u_9 - s_{68}s_{79}u_2u_5$
$s_{66}s_{70}u_3u_9 - s_{63}s_{68}u_2u_5$	$s_{73}s_{81}u_8u_9 - s_{76}s_{79}u_{12}u_5$	$s_{70}s_{81}u_8u_9 - s_{63}s_{76}u_{12}u_5$
$s_{66}s_{73}u_8u_9 - s_{68}s_{79}u_{12}u_5$	$s_{66}s_{70}u_8u_9 - s_{63}s_{68}u_{12}u_5$	$s_{76}s_{79}u_{15}u_2 - s_{73}s_{81}u_{17}u_3$
$s_{68}s_{79}u_{15}u_2 - s_{66}s_{73}u_{17}u_3$	$s_{63}s_{76}u_{15}u_2 - s_{70}s_{81}u_{17}u_3$	$s_{63}s_{68}u_{15}u_2 - s_{66}s_{70}u_{17}u_3$
$s_{76}s_{79}u_{12}u_{15} - s_{73}s_{81}u_8u_{17}$	$s_{68}s_{79}u_{12}u_{15} - s_{66}s_{73}u_8u_{17}$	$s_{63}s_{76}u_{12}u_{15} - s_{70}s_{81}u_8u_{17}$
$s_{63}s_{68}u_{12}u_{15} - s_{66}s_{70}u_8u_{17}$		

C.3 The invariant family $\mathcal{X} \to \mathcal{T}^{\mathcal{G}}$

The equations are included in Figure C.1. The variables $\{s_1, \cdots, s_6\}$ correspond to the orbits, in the same order as listed above.

C.4 The equatorial sphere Δ_{eq}

The equatorial sphere is a simplicial complex on the vertex set

$$\{2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 17\}.$$

The rank-constant elements of $\mathcal{L}_{3,6}$ correspond to the remaining vertices, namely $\{1, 4, 18, 19, 20\}$. The maximal faces of the sphere Δ_{eq} are those in the table below:

.

(14, 5, 8, 2)	(5, 17, 8, 2)	(14, 5, 2, 10)	(5, 17, 2, 10)	(14, 7, 8, 2)	(16, 7, 8, 2)
(14, 7, 2, 10)	(16, 7, 2, 10)	(16, 17, 8, 2)	(16, 17, 2, 10)	(14, 7, 3, 6)	(16, 7, 3, 6)
(9, 15, 3, 6)	(16, 9, 3, 6)	(14, 15, 3, 6)	(14, 7, 12, 3)	(16, 7, 12, 3)	(9, 15, 12, 3)
(16, 9, 12, 3)	(14, 15, 12, 3)	(14, 5, 8, 6)	(5, 17, 8, 6)	(14, 5, 11, 6)	(5, 17, 11, 6)
(14, 5, 11, 10)	(5, 17, 11, 10)	(14, 7, 8, 6)	(16, 7, 8, 6)	(16, 17, 8, 6)	(9, 15, 11, 6)
(16, 9, 11, 6)	(14, 15, 11, 6)	(16, 17, 11, 6)	(14, 7, 12, 10)	(16, 7, 12, 10)	(9, 15, 11, 10)
(16, 9, 11, 10)	(9, 15, 12, 10)	(16, 9, 12, 10)	(14, 15, 11, 10)	(16, 17, 11, 10)	(14, 15, 12, 10)

C.5 Indescribable equations

C.5.1 X_{1345}

The fiber X_{1345} has two components. One of them is described by the equations below. The equations of the other component is obtained by a permutation of the coordinate functions of $\mathbb{P}^{19}$.

<table>
<tr><td>x_{12}</td><td>x_9</td><td>x_3</td><td>x_{15}</td></tr>
<tr><td>$x_2x_{16} - x_7x_{17}$</td><td>$x_8x_{10} - x_5x_{20}$</td><td>$x_6x_{10} - x_{11}x_{20}$</td><td>$x_5x_7 - x_8x_{14} - x_2x_{19}$</td></tr>
<tr><td>$x_1x_7 - x_{14}x_{16}$</td><td>$x_5x_6 - x_8x_{11}$</td><td>$x_1x_2 - x_{14}x_{17}$</td><td>$x_2x_6 - x_4x_8 - x_{17}x_{20}$</td></tr>
<tr><td>$x_1x_8 - x_5x_{16} + x_{17}x_{19}$</td><td>$x_2x_6 - x_4x_8 - x_{17}x_{20}$</td><td>$x_4x_5 - x_2x_{11} + x_{10}x_{17}$</td><td></td></tr>
<tr><td colspan="4">$x_7x_{11} - x_6x_{14} - x_{10}x_{16} - x_4x_{19} + x_1x_{20}$</td></tr>
</table>

$x_{18}x_{20}s_1^2s_2s_5s_6 - x_4x_7s_2s_4 + x_2x_3$	$x_1x_{20}s_1^2s_2s_3s_6^2 - x_4x_{19}s_2s_3s_4^2 - x_6x_{14}s_1s_2 + x_3x_5$
$x_{18}x_{20}s_1^2s_2s_5s_6 - x_4x_7s_2s_4 + x_2x_3$	$x_1x_{20}s_1^2s_2s_3s_6^2 - x_4x_{19}s_2s_3s_4^2 - x_6x_{14}s_1s_2 + x_3x_5$
$-x_{17}x_{20}s_1s_6 + x_4x_8s_4 + x_2x_6$	$-x_2x_{19}s_3s_4 + x_8x_{14}s_1 + x_5x_7$
$x_{16}x_{20}s_1s_2s_3s_6 + x_6x_7s_2 + x_3x_8$	$x_1x_{20}s_1^2s_2s_3s_6^2 - x_4x_{19}s_2s_3s_4^2 - x_{10}x_{16}s_1s_2 + x_2x_9$
$x_1x_{13}s_1^2s_2s_5s_6 - x_{11}x_{19}s_2s_4 + x_5x_9$	$-x_3x_{19}s_3s_4 + x_{12}x_{16}s_1 + x_7x_9$
$x_6x_{19}s_2s_3s_4 + x_{13}x_{16}s_1s_2s_5 + x_8x_9$	$-x_{15}x_{20}s_1s_6 + x_4x_{12}s_4 + x_3x_{10}$
$-x_4x_{13}s_4s_5 + x_{11}x_{20}s_6 + x_6x_{10}$	$-x_5x_{20}s_3s_6 + x_2x_{13}s_5 + x_8x_{10}$
$-x_4x_5s_3s_4 + x_{10}x_{17}s_1 + x_2x_{11}$	$-x_4x_9s_3s_4 + x_6x_{15}s_1 + x_3x_{11}$
$-x_{13}x_{17}s_1s_5 + x_5x_6s_3 + x_8x_{11}$	$x_{14}x_{20}s_1s_2s_3s_6 + x_7x_{10}s_2 + x_2x_{12}$
$x_{10}x_{19}s_2s_3s_4 + x_{13}x_{14}s_1s_2s_5 + x_5x_{12}$	$-x_9x_{20}s_3s_6 + x_3x_{13}s_5 + x_6x_{12}$
$x_{19}x_{20}s_2s_3^2s_4s_6 - x_7x_{13}s_2s_5 + x_8x_{12}$	$-x_{13}x_{15}s_1s_5 + x_9x_{10}s_3 + x_{11}x_{12}$
$-x_1x_{12}s_1s_6 + x_{15}x_{19}s_4 + x_9x_{14}$	$x_4x_{14}s_2s_3s_4 + x_{10}x_{18}s_1s_2s_5 + x_2x_{15}$
$x_1x_{10}s_1s_2s_3s_6 + x_{11}x_{14}s_2 + x_5x_{15}$	$-x_{12}x_{18}s_1s_5 + x_3x_{14}s_3 + x_7x_{15}$
$-x_1x_8s_1s_6 + x_{17}x_{19}s_4 + x_5x_{16}$	$-x_{18}x_{19}s_4s_5 + x_1x_7s_6 + x_{14}x_{16}$
$-x_1x_3s_3s_6 + x_9x_{18}s_5 + x_{15}x_{16}$	$x_4x_{16}s_2s_3s_4 + x_6x_{18}s_1s_2s_5 + x_3x_{17}$
$-x_8x_{18}s_1s_5 + x_2x_{16}s_3 + x_7x_{17}$	$x_1x_6s_1s_2s_3s_6 + x_{11}x_{16}s_2 + x_9x_{17}$
$-x_1x_2s_3s_6 + x_5x_{18}s_5 + x_{14}x_{17}$	$x_1x_4s_2s_3^2s_4s_6 - x_{11}x_{18}s_2s_5 + x_{15}x_{17}$

$$x_1x_{20}s_1^2s_3^2s_6^2 - x_4x_{19}s_3^2s_4^2 - x_{13}x_{18}s_1^2s_5^2 - x_6x_{14}s_1s_3 - x_{10}x_{16}s_1s_3 + x_7x_{11}$$
$$x_1x_{20}s_1s_2s_3^2s_6^2 - x_{13}x_{18}s_1s_2s_5^2 - x_6x_{14}s_2s_3 + x_8x_{15}$$
$$x_1x_{20}s_1s_2s_3^2s_6^2 - x_{13}x_{18}s_1s_2s_5^2 - x_{10}x_{16}s_2s_3 + x_{12}x_{17}$$

Figure C.1: Equations for the invariant family $\mathcal{X} \to \mathcal{T}^{\mathcal{G}}$.

$$
\begin{array}{l|l}
x_2x_3 - x_4x_7s_{65}u_8 + x_{18}x_{20}s_{68}s_{70}s_{77}u_5u_{11} & x_2x_6 + x_4x_8u_8 - x_{17}x_{20}s_{68}u_{11} \\
x_5x_7 + x_8x_{14}s_{63} - x_2x_{19}s_{81}u_{17} & x_6x_7s_{65} + x_3x_8 + x_{16}x_{20}s_{68}s_{65}s_{81}u_{11} \\
x_5x_9 + x_1x_{13}s_{66}s_{70}s_{77}u_3u_7 - x_{11}x_{19}s_{77}u_{15} & x_7x_9 + x_{12}x_{16}s_{66} - x_3x_{19}s_{79}u_{15} \\
x_8x_9 + x_{13}x_{16}s_{66}s_{65}u_3 + x_6x_{19}s_{82}s_{73}u_{17} & x_3x_{10} + x_4x_{12}u_{12} - x_{15}x_{20}s_{70}u_{11} \\
x_6x_{10} - x_4x_{13}u_{12}u_3 + x_{11}x_{20}u_{11} & x_8x_{10} + x_2x_{13}u_2 - x_5x_{20}s_{73}u_{11} \\
-x_4x_5s_{73}u_8 + x_2x_{11} + x_{10}x_{17}s_{68} & -x_4x_9s_{76}u_{12} + x_3x_{11} + x_6x_{15}s_{70} \\
x_5x_6s_{73} + x_8x_{11} - x_{13}x_{17}s_{68}u_2 & x_7x_{10}s_{74} + x_2x_{12} + x_{14}x_{20}s_{70}s_{74}s_{79}u_{11} \\
x_5x_{12} + x_{13}x_{14}s_{70}s_{77}u_3 + x_{10}x_{19}s_{74}s_{81}u_{17} & x_6x_{12} + x_3x_{13}u_3 - x_9x_{20}s_{76}u_{11} \\
x_8x_{12} - x_7x_{13}s_{74}u_2 + x_{19}x_{20}s_{74}s_{73}s_{81}u_{17}u_{11} & x_9x_{10}s_{76} + x_{11}x_{12} - x_{13}x_{15}s_{70}u_3 \\
-x_1x_{12}s_{66}u_7 + x_9x_{14} + x_{15}x_{19}u_{15} & x_4x_{14}s_{74}s_{79}u_{12} + x_2x_{15} + x_{10}x_{18}s_{68}s_{77}u_5 \\
x_1x_{10}s_{68}s_{77}s_{81}u_7 + x_{11}x_{14}s_{77} + x_5x_{15} & x_3x_{14}s_{79} + x_7x_{15} - x_{12}x_{18}s_{66}u_9 \\
-x_1x_8s_{63}u_7 + x_5x_{16} + x_{17}x_{19}u_{17} & x_1x_7u_7 + x_{14}x_{16} - x_{18}x_{19}u_{17}u_5 \\
-x_1x_3s_{79}u_7 + x_{15}x_{16} + x_9x_{18}u_9 & x_4x_{16}s_{74}s_{81}u_{12} + x_3x_{17} + x_6x_{18}s_{70}s_{77}u_5 \\
x_2x_{16}s_{81} + x_7x_{17} - x_8x_{18}s_{63}u_5 & x_1x_6s_{70}s_{82}s_{79}u_7 + x_{11}x_{16}s_{82} + x_9x_{17} \\
-x_1x_2s_{81}u_7 + x_{14}x_{17} + x_5x_{18}u_5 & x_1x_4s_{74}s_{79}s_{81}u_{12}u_7 + x_{15}x_{17} - x_{11}x_{18}s_{77}u_5
\end{array}
$$

$$
\begin{array}{l}
x_3x_5 - x_6x_{14}s_{63}s_{65} - x_4x_{19}s_{74}s_{81}u_{12}u_{17} + x_1x_{20}s_{68}s_{70}s_{77}s_{81}u_{11}u_7 \\
x_2x_9 - x_{10}x_{16}s_{66}s_{74} - x_4x_{19}s_{82}s_{73}u_8u_{17} + x_1x_{20}s_{68}s_{70}s_{82}s_{79}u_{11}u_7 \\
x_7x_{11} - x_6x_{14}s_{63}s_{73} - x_{10}x_{16}s_{68}s_{81} - x_{13}x_{18}s_{63}s_{68}u_2u_5 - x_4x_{19}s_{73}s_{81}u_8u_{17} \\
\qquad + x_1x_{20}s_{68}s_{70}s_{79}s_{81}u_{11}u_7 \\
-x_6x_{14}s_{65}s_{79} + x_8x_{15} - x_{13}x_{18}s_{68}s_{77}u_2u_5 + x_1x_{20}s_{68}s_{65}s_{79}s_{81}u_{11}u_7 \\
-x_{10}x_{16}s_{74}s_{81} + x_{12}x_{17} - x_{13}x_{18}s_{70}s_{77}u_3u_5 + x_1x_{20}s_{70}s_{74}s_{79}s_{81}u_{11}u_7
\end{array}
$$

Figure C.2: The equations of the family $\mathcal{X} \to \mathcal{T}'$.

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