Compute invariants using Macaulay2

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We describe a recipe for finding invariant rings in Macaulay2 [2], using results from "Ideals, varieties and algorithms" [1].

1 Preliminaries

Suppose a matrix group G acts on a affine space $\mathbb{A}^n$, by actions fixing the origin. If $M \in \mathrm{GL}_n(k)$ acts on $P \in \mathbb{A}^n$ by $P \mapsto MP$, the corresponding action on the coordinate rings are given by $x_i \mapsto A^T x_i$.

Thus for example, if we look at C_4, rotating the plane by 90 degrees counterclockwise, that is

$$P = (p_1, p_2)^t \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (p_1, p_2)^t,$$

the corresponding k-algebra morphism is given by $x \mapsto -y$ and $y \mapsto x$.

We will return to this example in the next section.

Now, by definition, the invariants of $k[x_1, \ldots, x_n]$ are all the polynomials left unchanged by the action of G. It isn't even clear from the outset that the ring $k[x_1, \ldots, x_n]^G$ is finitely generated! Luckily this is true if G is a finite group, for example (or more generally, a linearly reductive group).

The crucial thing is the existence of the *Reynolds operator*. This is a linear map $k[x_i] \to k[k_i]$ which is a projection onto the G-invariants. It is given by "averaging":

$$R_G(f(x)) = \frac{1}{|G|} \sum_{A \in G} f(A \cdot x) = \frac{1}{|G|} \sum_{A \in G} A \cdot f(x). \tag{1}$$

In fact, we have the following theorem of Emmy Noether:

Theorem 1.1. *Let $G \subset \mathrm{GL}(n)$ be a finite matrix group. Let R_G be the Reynolds operator. Then we have*

$$k[x_1, \ldots, x_n]^G = k[R_G(x^\beta) \mid |\beta| \leq |G|].$$

This is *the* theorem needed to compute the ring of invariants. Note however that *many* monomials need to be computed. If G has order m then we need to compute the Reynolds operator $\binom{n+m}{m}$ times [1]. However, with computer, redundancy is never a problem in small examples.

2 Doing this

Since we have chosen to use `Macaulay2`, the first problem is how to represent a group. I've chosen to represent it as a list of ring maps $k[x_i] \to k[x_i]$.

Thus for the example of C_4 acting on $k[x, y]$ can be represented in the following way:

```
r1 = map(R,R,{-y,x})
r2 = r1 * r1
r3 = r2 * r1
r4 = r3 * r1
G = {r1,r2,r3,r4}
```

The Reynolds operator can be coded as follows:

```
reynolds = method()
reynolds(RingElement, List) := (f,G) (
    card := #G;
    r = (sum apply(G, g -> g f))/card;
    r
    )
```

All monomials of degree less than $|G|$ can be found by $basis(1, m, R)$. We compute Reynolds on all monomials:

```
monomials = flatten entries basis(1,4,R)
rlist = unique apply(monomials, m -> reynolds(m,G))
```

We get four invariant polynomials:

[1] This follows from the identity $\sum_{d=0}^{m} \binom{n+d}{d} = \binom{n+m+1}{m}$.

```
         1 2   1 2  1 4   1 4  1 3    1   3   2 2    1 3    1   3
o33 = {0, -x  + -y , -x  + -y , -x y - -x*y , x y , - -x y + -x*y }
          2     2    2     2    2      2               2      2
```

```
o33 : List
```

Not all of them are necessary however. We can get a minimal presentation by the wonderful command `minimalPresentation` in `Macaulay2`.

We write this as follows:

```
S = QQ[z_0..z_(#rlist-1)]
phi = map(R,S, rlist)
A = S/ker phi
minimalPresentation A
```

The outut is the following:

```
i65 : minimalPresentation A

        QQ[z , z , z ]
            1   4   5
o65 = -----------------
        2       2     2
      2z z  - 2z  - 2z
        1 4     4     5

o65 : QuotientRing
```

This means that the invariant ring is generated by the second, fifth and sixth (counting starts at zero) element of the Reynolds list. These are:

```
i68 : {rlist#1,rlist#4,rlist#5}

       1 2   1 2   2 2    1 3    1   3
o68 = {-x  + -y , x y , - -x y + -x*y }
       2     2            2      2
```

Replacing with scalar multiples, we conclude that

$$k[x,y]^G = k[x^2+y^2, x^2y^2, x^3y-xy^3].$$

References

[1] David Cox, John Little, and Donal O'Shea. *Ideals, varieties, and algorithms*. Undergraduate Texts in Mathematics. Springer, New York, third edition, 2007. An introduction to computational algebraic geometry and commutative algebra.

[2] Daniel R. Grayson and Michael E. Stillman. Macaulay2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/.