University of Oslo

MAT2000 Project

The Banach-Tarski Paradox

Author:
Fredrik Meyer

Supervisor:
Nadia S. Larsen

Abstract

In its weak form, the Banach-Tarski paradox states that for any ball in $\mathbb{R}^3$, it is possible to partition the ball into finitely many pieces, reassemble them using rotations only, producing two new balls of the exact size as the original ball. In its strong form, the paradox states that for any two bounded sets $A, B \in \mathbb{R}^3$ with non-empty interior, it is possible to partition A, move the pieces around, and end up with B.

For a paradoxical decomposition of the sphere, it can be shown that 4 pieces actually is enough. This short paper aims to prove the Banach-Tarski paradox in its weak and strong forms, together with the just-mentioned lower bound on the number of pieces in the decomposition.

May 30, 2010

1 Introduction

The notion of infinity has always been well-known to create seemingly paradoxical situations. One famous example is that the cardinality of $\mathbb{Z}$ equals the cardinality of $2\mathbb{Z}$, even though the one is a proper subset of the other. A more leisure-like example is the following:

Lemma 1.1 (Hotel Paradox)**.** *Suppose you have a hotel with an infinite number of rooms. Suppose further that all rooms are occupied. Then, if a new guest arrives, there is still room for him.*

Proof. Let the guest in room number n be transferred to room number $n+1$. This "shifting to infinity" leaves room number 1 vacant. $\square$

We state the main goal of this paper:

Theorem 1.2 (The Strong Banach-Tarski Paradox)**.** *For any two bounded sets A, B with non-empty interior it is possible to partition A into finitely many pieces, move the pieces around, and end up with B.*

Even though the Banach-Tarski paradox may sound unbelievable, it hardly is. The ideas used in the proofs leading to the theorem, all depend on basically the same idea as in the proof of the Hotel Paradox. This easier proof shows the main idea behind several of the proofs leading to the paradox:

Theorem 1.3. *The unit circle with one point removed, $S^1 \backslash \{pt\}$, may be partitioned into two pieces A, B, such that, after rotating B to B', we have $A \cup B' = S^1$.*

Proof. Let us identify the unit circle with $\{z \in \mathbb{C} | |z| = 1\}$. Without loss of generality, one may assume that the point removed is 1. Our aim is to produce some infinite set which, like in the Hotel Paradox, let us "shift towards infinity". Thus, let $B = \left\{e^{in} | n \in \mathbb{N}\right\}$. Since 2π is irrational, this set is infinite (and in particular, all members of the set are distinct). Let $A = (S^1 \backslash \{pt\}) \backslash B$. It is clear that $S^1 \backslash \{1\} = A \cup B$. Now, rotate B using the isometry $\rho(z) = e^{-iz}$. This generates the set $B' = \left\{e^{in} | n \in \mathbb{N} \cup \{0\}\right\}$. Thus $S^1 = A \cup B'$. $\square$

Notice the "shifting towards infinity" technique. The reader should amuse herself looking for where the same idea is used.

That being said, the Banach-Tarski paradox is not uninteresting - far from it. To quote Wagon [1]: "*Ideas arising from the Banach-Tarski Paradox have become the foundation of a theory of finitely additive measures, a* *theory that involves much interplay between analysis (measure theory and linear functionals), algebra (combinatorial group theory), geometry (isometry groups), and topology (locally compact topological groups).*" Finally, one must remember that although $\mathbb{R}^3$ is much similar to the space we live and breathe in, they are fundamentally different: $\mathbb{R}^3$ has uncountable cardinality, but our "space" most likely have finite cardinality (thus rendering miracles such as feeding the four thousand highly improbable).

The reader is assumed to know some linear algebra and some basic group theory. The definitions of countability and uncountability will also be used.

2 The Banach-Tarski Paradox

Recall that a group G is said to *act* on a set X if there is a bijection $G \times X \to X$ such that $g_1 g_2 \cdot x = g_1 \cdot (g_2 \cdot x)$ and $1 \cdot x = x$, for $g_1, g_2 \in G$ where 1 denotes the identy of G.

To add some substance to the word "paradoxical", we state its definition:

Definition 2.1. *Let G be a group acting a non-empty set X. Suppose there is a set $E \subseteq X$ such that there exist pairwise disjoint subsets $A_1, \ldots, A_n, B_1 \ldots, B_m$ of E and $g_1, \ldots, g_n, h_1, \ldots, h_m \in G$ such that*

$$E = \bigcup_{j=1}^{n} g_j \cdot A_j = \bigcup_{j=1}^{m} h_j \cdot B_j.$$

We say that E is G-paradoxical.

Note that if H is a subgroup of G, and a set X is H-paradoxical, then X is automatically G-paradoxical.

Every group acts naturally on itself by left translation. We will say that a group G is paradoxical if it is G-paradoxical when the action is left translation. Our first example of a paradoxical set will be important to us:

Lemma 2.2. *The free group $\mathbb{F}_2$ of order two is paradoxical.*

Proof. Let a, b be the generators of $\mathbb{F}_2$. Let $W(\rho)$ be the set of all words in $\mathbb{F}_2$ beginning with ρ from left. Then $W(a) \cup W(a^{-1}) \cup W(b) \cup W(b^{-1}) \cup \{e\} = \mathbb{F}_2$ and these sets are pairwise disjoint. Then $\mathbb{F}_2 = W(a) \cup aW(a^{-1})$. For if $w \notin W(a)$, then $a^{-1}w \in W(a^{-1})$, so $w = a(a^{-1}w) \in aW(a^{-1})$. Similarly, we have $\mathbb{F}_2 = W(b) \cup bW(b^{-1})$. □

If a paradoxical group G act on a set X in a particularly nice way, one can "lift" the paradox from the group to the set:

Lemma 2.3. *If G is paradoxical and acts on X with no non-trivial fixed points, then X is G-paradoxical.*

Proof. Let A_i, B_i, g_i, h_i be as in Definition 2.1. Using the Axiom of Choice, choose a set M such that M contains exactly one element from each G-orbit in X; then $\{g \cdot M : g \in G\}$ is a disjoint partition of X: Certainly, $\cup_{g \in G} g \cdot M = X$, since M contains one element from each G-orbit. To prove disjointness, assume there exist $g, h \in G$ such that $g \cdot M \cap h \cdot M \neq \varnothing$. Then there are $x, y \in M$ such that $g \cdot x = h \cdot y$. Then $h^{-1}g \cdot x = y$, so x and y are in the same G-orbit, but by our choice of M, this means that $x = y$. So we have $h^{-1}g \cdot x = x$. Since G act on X with no non-trivial fixed points, this means that $h^{-1}g = 1$, and thus must be equal.

Now, let

$$A_j^* = \bigcup_{g \in A_j} \{g \cdot M\} \ (j = 1, \ldots, n)$$

and

$$B_j^* = \bigcup_{g \in B_j} \{g \cdot M\} \ (j = 1, \ldots, m)$$

Obviously, $A_1^*, \ldots, A_n^*, B_1^*, \ldots, B_m^*$ are disjoint subsets of X (since A_i and B_j are disjoint). We claim that $X = \cup_{j=1}^n g_j \cdot A_j^* = \cup_{j=1}^m h_j \cdot B_j$, proving that X is G-paradoxical. For suppose $x \in X$. Because M contains one element from each G-orbit, there exists a $g \in G$ such that $x \in g \cdot M$. Since G is paradoxical, we know that $g = g_j a_j$ for some $a_j \in A_j$. Thus $x \in g_j(a_j \cdot M)$, and since $a_j \cdot M \in A_j^*$, we see that $x \in \cup_{j=1}^n g_j \cdot A_j^*$. The case for the B_j^*'s is of course identical.

□

An immediate consequence of the preceding lemma is the following:

Corollary 2.4. *If $\mathbb{F}_2$ acts on X with no non-trivial fixed points, then X is $\mathbb{F}_2$-paradoxical.*

Recall that $SO(3)$ is the group of rotations of the sphere, or equivalently, the group of 3×3-matrices A such that $\det A = 1$. Our next result will turn out to be the "recipe" for any paradoxical decomposition.

Lemma 2.5. *There are rotations A and B about lines through the origin in $\mathbb{R}^3$ generating a subgroup of $SO(3)$ isomorphic to $\mathbb{F}_2$, the free group of two generators.*

Proof. Let

$$A^{\pm} = \begin{bmatrix} \frac{1}{3} & \mp\frac{2\sqrt{2}}{3} & 0 \\ \pm\frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B^{\pm} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & \mp\frac{2\sqrt{2}}{3} \\ 0 & \pm\frac{2\sqrt{2}}{3} & \frac{1}{3} \end{bmatrix}$$

be our two rotations. Now, let w be a reduced word in $A^{\pm}, B^{\pm}$ which is not the empty word I. We claim that w cannot act as identity on $\mathbb{R}^3$, thus proving that $\langle A, B\rangle \simeq \mathbb{F}_2$, where $\langle A, B\rangle$ is the subgroup of $SO(3)$ generated by A, B. Note first that, without loss of generality, we may assume that w ends in $A^{\pm}$ (for if w were the empty word, then conjugation by $A^{\pm}$ will not alter this). We claim that

$$w \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{3^k} \begin{bmatrix} a \\ b\sqrt{2} \\ c \end{bmatrix}$$

where $a, b, c \in \mathbb{Z}$ and $3 \nmid b$ and k is the length of the word w, in which case w cannot possibly act as identity on $\mathbb{R}^3$. We will prove this by induction on k. The base case $k = 1$ is simple. By assumption then, $w = A^{\pm}$:

$$w \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \mp\frac{2\sqrt{2}}{3} & 0 \\ \pm\frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{\pm 2\sqrt{2}}{3} \\ 0 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ \pm 2\sqrt{2} \\ 0 \end{bmatrix}.$$

Certainly, $k = 1$ is fine. Now, let $w = A^{\pm}w'$ or $w = B^{\pm}w'$ where

$$w' \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{3^{k-1}} \begin{bmatrix} a' \\ b'\sqrt{2} \\ c' \end{bmatrix},$$

$a', b', c' \in \mathbb{Z}$ and b' is not divisible by 3. An easy calculation shows that

$$w \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{3^{k-1}} \overbrace{\begin{bmatrix} \frac{1}{3} & \mp\frac{2\sqrt{2}}{3} & 0 \\ \pm\frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{A^{\pm}} \begin{bmatrix} a' \\ b'\sqrt{2} \\ c' \end{bmatrix} = \frac{1}{3^{k-1}} \begin{bmatrix} \frac{1}{3}a' \mp \frac{1}{3}4b' \\ \pm a'\frac{2\sqrt{2}}{3} + \frac{1}{3}b'\sqrt{2} \\ c' \end{bmatrix}$$

$$= \frac{1}{3^k} \begin{bmatrix} a' \mp 4b' \\ \sqrt{2}(b' \pm 2a') \\ 3c' \end{bmatrix} = \frac{1}{3^k} \begin{bmatrix} a \\ b\sqrt{2} \\ c \end{bmatrix}.$$

And similarly for $w = B^{\pm}w'$. We have:

$$\begin{cases} a = a' \mp 4b', b = b' \pm 2a', c = 3c' \text{ if } w = A^{\pm}w' \\ a = 3a', b = b' \mp 2c', c = c' \pm 4b' \text{ if } w = B^{\pm}w' \end{cases} \tag{1}$$

Obviously, $a, b, c \in \mathbb{Z}$. We must show that $3 \nmid b$, and we will be done:

Case 1 $w = A^{\pm}B^{\pm}v$ (where possibly $v = I$). We then have

$$w \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = A^{\pm}B^{\pm}v \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

so that by (1), we have $b = b' \pm 2a' = b' \pm 6a''$. Since, by assumption, $3 \nmid b'$, it follows that $3 \nmid b$.

Case 2 $w = B^{\pm}A^{\pm}v$. Proven as Case 1.

Case 3 $w = A^{\pm}A^{\pm}v$. By assumption,

$$v \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{3^{k-2}} \begin{bmatrix} a'' \\ b''\sqrt{2} \\ c'' \end{bmatrix}$$

where $a, b, c \in \mathbb{Z}$ and $3 \nmid b''$. By (1) it follows that

$$b = b' \pm 2a' = b' \pm 2(a'' \mp 4b'') = b' + b'' \pm 2a'' - 9b'' = 2b' - 9b''$$

so $3 \nmid b$.

Case 4 $w = B^{\pm}B^{\pm}v$. This case is treated as above.

□

Our weak goal is to show that S^2 is $SO(3)$-paradoxical. That is, you can partition the sphere such that rotations on the pieces produce two new copies of S^2. The next result shows that this almost can be done.

Lemma 2.6 (Hausdorff Paradox)**.** *There is a countable subset D of S^2 such that $S^2\backslash D$ is $SO(3)$-paradoxical.*

Proof. Let A and B be as in the preceding proof. Let $G = \langle A, B\rangle \simeq \mathbb{F}_2$, so G is paradoxical. Since A and B are rotations about the origin, each $w \in G\backslash\{I\}$ has exactly two fixed points. Let D be the collection of all points in S^2 that are fixed by some $w \in G\backslash\{I\}$. Since G is countable, so is D. Clearly, G acts on $S^2\backslash D$ without non-trivial fixed points: for if $p \in S^2\backslash D$ and $g \in G$ and $g \cdot p = p$, we would have $p \in D$ which is impossible. It follows by Lemma 2.3 that $S^2\backslash D$ is G-paradoxical. Since $G \leq SO(3)$, we conclude that $S^2\backslash D$ is $SO(3)$-paradoxical. □

Any rectangle can be cut in two and moved around to yield an isosceles triangle. We say that the rectangle and the isosceles triangle are *equidecomposable.* This notion can be generalized to sets which are acted upon by groups.

Definition 2.7. *Suppose G is a group acting on a set X, and that $A, B \subseteq X$. Then A and B are called G-equidecomposable (we write $A \sim_G B$) if there exist subsets A_i of A and B_i of B such that*

$$A = \bigcup_{i=1}^{n} A_i = A,\ B = \bigcup_{i=1}^{n} B_i, \tag{2}$$

and $A_i \cap A_j = \varnothing = B_i \cap B_j$ for $i \neq j$, and $g_i \in G$ such that $g_i \cdot A_i = B_i$ for each $i \in \{1, 2, \ldots, n\}$.

When it is clear from the context which group we are talking about, we will write $A \sim B$ in place of $A \sim_G B$. Later, we will use the notation $A \sim_n B$ when A and B are equidecomposable using n pieces. It will be important for us to be sure that $\sim_G$ actually is an equivalence relation, and thus partitions $\mathcal{P}(X)$, the power set of X.

Lemma 2.8. *$\sim_G$ is an equivalence relation on the subsets of $\mathcal{P}(X)$.*

Proof. Obviously, $A \sim_G A$. Just let the identity of G act on A itself. So $\sim_G$ is reflexive. Now, assume $A \sim_G B$. We must show that also $B \sim_G A$. Since $A \sim_G B$, there exists a partition of A and B such that $g_i \cdot A_i = B_i$ for each cell in the partition. But then we have that $A_i = g_i^{-1}(g_i \cdot A_i) = g_i^{-1} \cdot B_i$ for each cell in the partition. Thus $B \sim_G$ A. So $\sim_G$ is symmetric.

To show transitivity, assume $A \sim_G B$ and $B \sim_G C$. First, observe that if $A \cap B = \varnothing$, then $gA \cap gB = \varnothing$. For if not, there would exist $a \in A$ and $b \in B$ such that $ga = gb$, but this implies that $a = b$, which is impossible (since $A \cap B = \varnothing$). Now, since $A \sim_G B$, there is a partition of $A = \cup_{i=1}^n A_i$ and $B = \cup_{i=1}^n B_i$ such that $g_i A_i = B_i$ for $1 \leq i \leq n$ and $g_i \in G$. And since $B \sim_G C$ there is a partition of $B = \cup_{j=1}^m B^j$ and $C = \cup_{j=1}^m C_j$ such that $h_j B^j = C_j$ for $1 \leq j \leq m$ and $h_i \in G$. Now, observe that since

$$B_i = (\bigcup_{j=1}^{m} B^j) \bigcap B_i = \bigcup_{j=1}^{m} (B_i \bigcap B^j),$$

we have

$$g_iA_i = \cup_{j=1}^m (B_i \cap B^j),$$

which, by our observation, implies that

$$\left\{g_i^{-1}(B_i \cap B^j) | 1 \leq i \leq n, 1 \leq j \leq m\right\}$$

is a disjoint partition of A. Thus, since $B^j = \cup_{i=1}^n (B_i \cap B^j)$, it follows that $h_j g_i (g_i^{-1} \cup_{i=1}^n (B_i \cap B^j)) = C_j$. We conclude that $A \sim_G C$. $\square$

Notice that in the above proof, we used at most nm pieces. An easy - but nonetheless very important - observation about this relation is the following:

Lemma 2.9. *If $A_1 \cap A_2 = \varnothing = B_1 \cap B_2$ with $A_1 \sim B_1$ and $A_2 \sim B_2$, then $A_1 \cup A_2 \sim B_1 \cup B_2$.*

Proof. $A_1 \sim B_1$ means that $B_1 = \cup g_i A_1^i$ for some partition of A_1 and $g_i \in G$, and similarly $B_2 = \cup g_j A_2^j$. Then, since $A_1 \cap A_2 = \varnothing$, we can use the same partition of $A_1 \cup A_2$ as in A_1, A_2. The result follows. $\square$

Equidecomposability makes it easier to determine if a set is paradoxical:

Lemma 2.10. *Suppose G is a group acting on a set X and that E and E' are G-equidecomposable subsets of X. If E is G-paradoxical, so is E'.*

Proof. Assume E is G-paradoxical. Then there are $A_i, B_i \subset E$ and g_i, h_i as in Definition 2.1 where

$$A = \bigcup_{j=1}^n A_j \text{ and } B = \bigcup_{j=1}^m B_j$$

Where $\cup_{j=1}^n g_i \cdot A_i = E$ and $\cup_{j=1}^m h_j \cdot B_j = E$. Then $A \sim_G E$ and $B \sim_G E$. Since $\sim_G$ is an equivalence relation, it is transitive, so since $E \sim_G E'$, we have $A \sim_G E'$ and $B \sim_G E'$. But from the definition of equidecomposability, this implies that there exist a partition A_j of A and group elements k_i such that $E' = \cup_j k_J \cdot A_j$, and also a partition B_j of B and group elements k_j' such that $E' = \cup_j k_J' \cdot B_j$. But this means that E' is paradoxical. $\square$

The next result shows that if you remove any countable collection of points from the sphere, you can always restore it using rotations.

Lemma 2.11. *If D is a countable subset of S^2, then $S^2 \sim_{SO(3)} S^2 \backslash D$.*

Proof. Let L be a line through the origin such that $L \cap D = \varnothing$. Now, define as follows:

$$W = \left\{ \theta \in [0, 2\pi) \middle| \left\{ \begin{array}{l} \rho \text{ is a rotation through } L \text{ by the angle } n\theta \\ \text{for some integer } n \text{ such that if} \\ x \in D \Rightarrow \rho \cdot x \in D \end{array} \right. \right\}$$

Since D is countable, so is W. It follows that there is an element $\alpha \in [0, 2\pi) \backslash W$. Let ρ be the rotation about L through the angle α. By the definition of W, it follows that $\rho^n \cdot D \cap D = \varnothing$ for all integers $n \in \mathbb{N}$. It follows that

$$\rho^n \cdot D \cap \rho^m \cdot D = \varnothing \ (n, m \in \mathbb{N}, n \neq m). \tag{3}$$

Now, let $\bar{D} = \cup_{n=0}^{\infty} \rho^n \cdot D$. Notice that, by (3), $\rho \cdot \bar{D} \backslash \bar{D} = D$. Then

$$S^2 = \bar{D} \cup (S^2 \backslash \bar{D}) \sim \rho \cdot \bar{D} \cup (S^2 \backslash \bar{D}) = S^2 \backslash D,$$

as desired. Notice the use of Lemma 2.9. □

Corollary 2.12. *S^2 is SO(3)-paradoxical.*

Proof. Combining Lemmas 2.6, 2.10 and 2.11 immediately gives us the result: Lemma 2.6 tells us that there exists a countable subset $D \subset S^2$ such that $S^2 \backslash D$ is $SO(3)$-paradoxical. Lemma 2.11 tells us that $S^2 \backslash D \sim_{SO(3)} S^2$ and combined with Lemma 2.10, we have that S^2 is $SO(3)$-paradoxical. □

Let O_3 denote the set of all Euclidean isometries preserving handedness, that is, the set of all translations and rotations. The weak Banach-Tarski paradox is a slightly stronger statement than Corollary 2.12.

Lemma 2.13 (Weak Banach-Tarski Paradox)**.** *Every closed ball in $\mathbb{R}^3$ is O_3-paradoxical.*

Proof. Without loss of generality, we prove only that $B_1[0, \mathbb{R}^3]$, the closed unit ball, is paradoxical. The proof for any other ball is identical but more messy. We show first that $B_1[0, \mathbb{R}^3] \backslash \{0\}$ is paradoxical.

Since S^2 is $SO(3)$-paradoxical, there are $A_1, \ldots, A_n, B_1, \ldots, B_m \subset S^2 \subset \mathbb{R}^3$ and $g_1, \ldots, g_n, h_1, \ldots, h_m \in SO(3)$ as in Definition 2.1. Let

$$A_j^* = \{tx | t \in (0, 1], x \in A_j\} \ \ (j = 1, \ldots, n)$$

and

$$B_j^* = \{tx | t \in (0,1], x \in B_j\} \ \ (j = 1, \ldots, m)$$

We note that $A_i^* \cap A_j^* = \varnothing$ when $i \neq j$ and that

$$B_1[0, \mathbb{R}^3] \setminus \{0\} = \bigcup_{j=1}^{n} g_j \cdot A_j^* = \bigcup_{j=1}^{m} h_j \cdot B_j^*$$

It follows from the definition of paradoxality that $B_1[0, \mathbb{R}^3] \setminus \{0\}$ is indeed paradoxical.

Now, let $x = (0, 0, \frac{1}{2}) \in B_1[0, \mathbb{R}^3] \setminus \{0\}$. Let ρ be a rotation of infinite order about a line through x not through the origin. Let $D = \{\rho^n \cdot 0 | n \in \mathbb{N}_0\}$. Obviously, $\rho \cdot D = D \setminus \{0\}$. Then, by earlier results:

$$B_1[0, \mathbb{R}^3] = D \cup (B_1[0, \mathbb{R}^3] \setminus D) \sim \rho \cdot D \cup (B_1[0, \mathbb{R}^3] \setminus D) = B_1[0, \mathbb{R}^3] \setminus \{0\}$$

It follows by Lemma 2.10 that $B_1[0, \mathbb{R}^3]$ is $SO(3)$-paradoxical, and $SO(3)$ is a subgroup of O_3. $\square$

An almost identical proof gives us the following:

Corollary 2.14. $\mathbb{R}^3$ *is paradoxical.*

We want however more surprising mathematics. On our road to the strong form of the Paradox, we need the following very important observation:

Lemma 2.15. *If $A \sim_G B$, then there is a bijection $f : A \to B$ such that $C \sim f(C)$ whenever $C \subseteq A$.*

Proof. Since $A \sim B$, there is a partition of $A = \cup_{i=1}^n A_i$ and a partition of $B = \cup_{i=1}^n B_i$ such that $g_i \cdot A_i = B_i$ for $g_i \in G$. Now, define $f : A \to B$ as follows:

$$f(x) = g_i \cdot x \text{ if } x \in A_i$$

Since $A \sim B$, this function is onto, and it is clearly $1-1$ since it is defined by group actions. Thus f is a bijection. To prove the latter part of the statement, assume $C \subseteq A$. Then $A_i \cap C$ gives us a partition of C. Since $f(C) = \cup_{i=1}^n g_i \cdot (A_i \cap C)$, it follows by the definition of equidecomposability that $C \sim f(C)$. $\square$

We define a relation as follows:

Definition 2.16. *Let G be a group acting on a set X, and let $A, B \subseteq X$. If $A \sim_G B_1$ where $B_1 \subseteq B$, we write $A \preccurlyeq_G B$.*

As with $\sim_G$, we will write $A \preccurlyeq B$ in place of $A \preccurlyeq_G B$ when this simplification does not cause any confusion.

Obviously, $\preccurlyeq_G$ is reflexive. To prove transitivity, assume $A \preccurlyeq B$ and $B \preccurlyeq C$. Then $A \sim B_1$ for some $B_1 \subseteq B$. By Lemma 2.15, $B_1 \sim f(B_1) \subseteq C$, so $A \sim B_1 \sim f(B_1) \subseteq C$. Thus $A \preccurlyeq C$.

We have, in fact, that if $A \preccurlyeq B$ and $B \preccurlyeq A$, then $A \sim B$. In other words, $\preccurlyeq$ is an antisymmetric binary relation. Thus $\preccurlyeq$ defines a partial ordering on the $\sim$-classes of $\mathcal{P}(X)$.

Theorem 2.17 (Banach-Schröder-Bernstein Theorem)**.** *Suppose G is a group acting on a set X. If $A, B \subseteq X$, $A \preccurlyeq_G B$, $B \preccurlyeq_G A$, then $A \sim_G B$.*

Proof. Let $f : A \to B_1(\subseteq B)$ and $g : B \to A_1(\subseteq A)$ be bijections as guarenteed by Lemma 2.15. Let $C_0 = A\backslash A_1$ and define inductively $C_{n+1} = gf(C_n)$ where $g \circ f : A \to A_1$, and let $C = \cup_{n=0}^{\infty} C_n$. We claim that $A\backslash C = g(B\backslash f(C))$. So assume $x \in B\backslash f(C)$. Then we can't have $g(x) \in C_n$ for any $n > 0$, for that would imply that $x \in f(C_{n-1})$. Thus $g(B\backslash f(C)) \subseteq A\backslash C$. Now, note that $A\backslash C = A_1\backslash C$. So assume $x \in A_1\backslash C$. Then there exists a $y \in B$ such that $g(y) = x$. If $y \in f(C)$, we would have $g(y) = x \in C$, which is impossible, so $y \in B\backslash f(C)$. Thus $A\backslash C \subset g(B\backslash f(C))$. So $A\backslash C = g(B\backslash f(C))$.

By our definition of g, this means that $A\backslash C \sim B\backslash f(C)$. In the same way, $C \sim f(C)$. So $A = (A\backslash C) \cup C \sim (B\backslash f(C)) \cup f(C) = B$ by Lemma 2.9. □

The Banach-Schröder-Berstein theorem gives us a characterization of paradoxical sets in terms of equidecomposability:

Lemma 2.18. *Let G be a group acting on a set X, and let $E \subseteq X$. Then E is G-paradoxical if and only if there exist disjoint A, B such that $A \cup B = E$ and $A \sim E \sim B$.*

Proof. We already know that if E is G-paradoxical, then we can find disjoint A, B such that $A \sim E \sim B$. It remains to show that we can find A', B' such that $A' \cup B' = E$ and such that the properties of A, B still holds.

Let $f : E \to A$ be a bijection as guaranteed by Lemma 2.15. Then, since $f(E\backslash B) \subset A$ and $f(E\backslash B) \sim E\backslash B$, we see that $E\backslash B \preccurlyeq A$, so $E\backslash B \preccurlyeq E$. And since $E \sim A \subset E\backslash B$, we have $E \preccurlyeq E\backslash B$. Thus, by the Banach-Schröder-Bernstein theorem, we have $E \sim E\backslash B$. So choose $A' = E\backslash B$ and $B' = B$.

The reverse implication is true by the definition of paradoxality. □

We have finally developed enough machinery to present the strong form of the paradox:

Theorem 2.19. *If A and B are bounded subsets of $\mathbb{R}^3$, each having non-empty interior, then $A \sim B$.*

Proof. By the Banach-Schröder-Bernstein theorem, it is enough to show that $A \preccurlyeq B$, for a symmetric argument produces the opposite relation.

Since A is bounded, we may choose a ball K such that $A \subseteq K$. Since B has non-empty interior, we may choose a ball L such that $L \subseteq B$. For some n large enough, K may be covered by n overlapping copies of L. That is, choose translations t_i such that $K \subset \cup_{i=1}^n t_i \cdot L$. Now, choose translations t_i' such that $t_i' \cdot L \cap t_j' \cdot L = \varnothing$ for $i \neq j$ and let $S = \cup_{i=1}^n t_i \cdot L$. Then, using the weak Banach-Tarski paradox to copy L n times, we get $L \sim S$, and in particular $S \preccurlyeq L$. Then

$$A \subset K \subset \cup_{i=1}^n t_i \cdot L \preccurlyeq S \preccurlyeq L \subset B,$$

which together with Lemma 2.15 establishes the claim. □

This completes our work on the existence of paradoxical decompositions.

3 Minimizing the number of pieces

The rest of this paper will focus on minimizing the number of pieces required for a paradoxical decomposition of the sphere. Although the paradox itself may be interesting enough, knowing how many pieces suffice is interesting because of the intuitively amazing low bound on this number. In fact, the number of pieces required for a paradoxical decomposition of the sphere is 4.

First, let us formally define the number r which we want to minimize:

Definition 3.1. *Suppose G acts on X and $E, A, B \subseteq X$. Then E is G-paradoxical using r pieces if $A \cap B = \varnothing$ with $A \cup B = E$ such that*

$$A \sim_m E \sim_n B$$

where $m + n = r$.

From Lemma 2.2 we remember that $\mathbb{F}_2$ is paradoxical, but in our proof we did not use all of $\mathbb{F}_2$ to demonstrate paradoxality. This is however possible, as stated by the following lemma:

Lemma 3.2. *Let $\mathbb{F}_2$ be the free group generated by a, b. Then we can find a partition $A_1, \ldots, A_4$ of $\mathbb{F}_2$ such that $A_1 \cup b \cdot A_2 = \mathbb{F}_2$ and $A_3 \cup a \cdot A_4 = \mathbb{F}_2$, meaning that $\mathbb{F}_2$ is paradoxical using this partition. Moreover, for any $w \in \mathbb{F}_2$, the partition can be chosen such that w is in the same cell as the identity 1 of $\mathbb{F}_2$.*

This is a simple proof found in [3].

Proof. Suppose w is a word starting with $\rho = a^{-1}$, say. The proof for any other ρ is identical (exchange a^{-1} anywhere it occurs with ρ). Define A_i as follows: (where $W(\rho)$ denotes the set of all words starting with ρ)

$$\begin{aligned} A_1 &= W(b) \\ A_2 &= W(b^{-1}) \\ A_3 &= W(a) \setminus \{a^n | n \in \mathbb{N}\} \\ A_4 &= W(a^{-1}) \cup \{a^n | n \in \mathbb{N}\} \cup \{1\} \end{aligned}$$

Now, in the same way is in the proof of Lemma 2.2, we have that $\mathbb{F}_2 = A_1 \cup b \cdot A_2$. Showing that $\mathbb{F}_2 = A_3 \cup a \cdot A_4$ is only slightly harder. It is clear that all powers of a are included in $a \cdot A_4$. So assume w is not a power of a. Then if $w \notin W(a)$, $a^{-1}w \in W(a^{-1})$. It follows that $w = a(a^{-1}w) \in a \cdot W(a^{-1})$. Thus $\mathbb{F}_2$ is paradoxical using four pieces.

It is also clear that $1, w \in A_4$. □

There is a four-piece analog to Lemma 2.3:

Lemma 3.3. *Let $\mathbb{F}_2$ act on X without non-trivial fixed points. Then X is $\mathbb{F}_2$-paradoxical using four piees.*

Proof. The proof is identical to that of Lemma 2.3 (observe that lifting a paradox to a set preserves the number of pieces). Just use the partition of $\mathbb{F}_2$ given in Lemma 3.2. □

If G acts on X and $x \in X$, let $\mathrm{Stab}(x) = \{\sigma \in G | \sigma \cdot x = x\}$. A key concept in proving the main result of this section is the following:

Definition 3.4. *An action of a group G on a set X is called locally commutative if $Stab(x)$ is commutative for every $x \in X$. Equivalently, if two elements of G have a common fixed point, then they commute.*

We have already met locally commutative actions. Take the group of rotations of the sphere S^2; if two rotations share a fixed point, they must share the same axis, and thus commute.

Lemma 3.5. *Let $\mathbb{F}_2$ act on X such that the action is locally commutative. Then X is $\mathbb{F}_2$-paradoxical using four pieces.*

Proof. This proof is long and hard, but we'll soon get our reward.

We will use Lemma 3.2 to partition X into $A_1^*, A_2^*, A_3^*, A_4^*$ satisfying $b \cdot A_2^* = A_2^* \cup A_3^* \cup A_4^*$ and $a \cdot A_4^* = A_1^* \cup A_2^* \cup A_4^*$, which suffices to prove the theorem. We will place each element of X in A_i^* orbit by orbit (possibly using different partitions of $\mathbb{F}_2$ for different orbits).

Note that all $\mathbb{F}_2$-orbits in X are either composed only of fixed points or of no non-trivial fixed points: assume $g \cdot x = x$ for $x \in X$ and $g \in \mathbb{F}_2$. If $y \in \mathbb{F}_2 x$, then $y = h \cdot x$ for some $h \in \mathbb{F}_2$. Then

$$hgh^{-1} \cdot y = hgh^{-1}(h \cdot x) = hg \cdot x = h \cdot x = y,$$

so y is a fixed point. For points in orbits consisting of no fixed points, the same assignment of points is straightforward. For each orbit, fix any representative x. Then each element y in the orbit may be uniquely written as $y = v \cdot x$ with $v \in \mathbb{F}_2$. Now, apply Lemma 3.2 to partition $\mathbb{F}_2$ having the additional property that 1 and w lie in the same cell of the partition. Place y in A_i^* if $v \in A_i \subset \mathbb{F}_2$.

For orbits consisting entirely of fixed points, we've got to work harder. Fix an orbit O. Let w be a word of minimal length fixing a point x in O. Let ρ be the leftmost letter in w. We see that w cannot end in ρ^{-1}, for if it did $\rho^{-1} w \rho$ would fix $\rho^{-1} \cdot x$ and be shorter than w. We claim that we can write any point $y \in O$ uniquely as $v \cdot x$ where v does not end in w or ρ^{-1}. Why? Let v be a minimal word such that $y = v \cdot x$. We see that v cannot end in w or w^{-1}, because that would contradict v being a word of minimal length (both w and w^{-1} fixes x). If however v ends in ρ^{-1}, we choose vw which doesn't end in w or ρ^{-1} (because ρ is the first letter in w).

We will show that the only elements fixing x are powers of w. This is where local commutativity comes in. For assume u fixes x (in addition to w). Then, by local commutativity, $wu = uw$. But since $\mathbb{F}_2$ is free, this means that $w = t^k$ and $u = t^l$ for some integers k, l and $t \in \mathbb{F}_2$ (else we would have a non-trivial relation on the letters of $\mathbb{F}_2$). It follows from the minimality of w that $|k| < |l|$. Using the division algorithm, we know that $l = kn + r$ for some integers k, r with $0 \leq r < |k|$. Thus

$$x = u \cdot x = t^l \cdot x = t^{kn+r} \cdot x = t^r (t^k)^n \cdot x = t^r \cdot x.$$

Again, using the minimality of w, this yields that $r = 0$, and u is thus a power of w, as claimed.

The uniqueness of the representation $y = v \cdot x$ follows easily: Assume $y = v \cdot x = u \cdot x$ are both representations of the desired form (v should not end in w, ρ^{-1}). Then $u^{-1}v \cdot x = x$, so one of $u^{-1}v$ or $v^{-1}u$ is a positive power of w. Assume, without loss of generality, that it is $u^{-1}v$; then $u^{-1}v = w^n$ for some integer n. Since w begins with ρ, u^{-1} must begin with ρ (contradicting our assumption that u did not end in ρ^{-1}), or u^{-1} cancel against v, meaning v ends in w (contradicting our assumption that v did not), and we conclude that $u = v$.

Having established that each $y \in O$ can be written uniquely as $y = v \cdot x$, we put y in A_i^* if $v \in A_i \subset \mathbb{F}_2$. To show that this assignment works, consider first the relation $b \cdot A_2^* \subseteq A_2^* \cup A_3^* \cup A_4^*$. Suppose $y \in A_2^*$. Then y has a representation $y = v \cdot x$ for an x in the orbit with $v \in A_2$. Now, consider $b \cdot y$. If $bv \cdot y$ is the correct representation of $b \cdot y$, then since $v \in A_2$ implies $bv \in A_2 \cup A_3 \cup A_4$, $b \cdot y$ is properly placed in $A_2^* \cup A_3^* \cup A_4^*$. But bv might possibly end in w or ρ^{-1}. So, assume bv ends in w. Since v does not end in w, bv must equal w, so $b \cdot y = w \cdot x = x$. Since $bv \in A_2 \cup A_3 \cup A_4$, so is w, and by our choice of partition of $\mathbb{F}_2$, so is 1. Since $1 \cdot x$ is the unique representation of x, this implies that x and hence $b \cdot y = bv \cdot x = w \cdot x = x$ lies in $A_2^* \cup A_3^* \cup A_4^*$. Assume now that bv ends in ρ^{-1}. Since v does not, v must equal 1. Then $y = x$, $bv = b$ and so $\rho^{-1} = b$, and from our choice of ρ, w begins with b^{-1}. Since $1 = v \in A_2$, by our choice of partition of $\mathbb{F}_2$, $w \in A_2$ also, hence $bw \in A_2 \cup A_3 \cup A_4$. Since $bw \cdot x$ is the unique representation of $b \cdot x$ (note that w begins with b^{-1}), we have $b \cdot y = b \cdot x \in A_2^* \cup A_3^* \cup A_4^*$. An identical treatment works for the other containments $b^{-1} \cdot (A_2^* \cup A_3^* \cup A_4^*) \subseteq A_2^*$, $a \cdot A_4^* \subseteq A_1^* \cup A_2^* \cup A_4^*$, and $a^{-1} \cdot (A_1^* \cup A_2^* \cup A_4^*) \subseteq A_3^*$, completing the proof. □

After having tortured ourselves through the preceding proof, we may proudly present the main result of this section. The sphere, S^2 is paradoxical using 4 pieces only, and if you ever try to improve that number, you won't succeed.

Theorem 3.6. *S^2 is $SO(3)$-paradoxical using four pieces, and the four cannot be improved.*

Proof. Since the action of rotations on the sphere is locally commutative (by the argument in the preceding paragraph), and $SO(3)$ have a subgroup isomorphic to $\mathbb{F}_2$, the first part of the claim follows from Lemma 3.5.

To demonstrate that a composition in four pieces cannot be improved, assume the contrary: that S^2 contains two disjoint subsets A, B such that $A \sim_m X \sim_n B$ where $m + n < 4$. At least one of m or n must equal 1. If

$m = 1$, then $S^2 = g(A)$ for some $g \in SO(3)$, whence $A = g^{-1}(S^2) = S^2$ and $B = \varnothing$, a contradiction. $\square$

And we stop here.

References

[1] Stan Wagon, *The Banach-Tarski Paradox*[1], Cambridge University Press, 1985.

[2] Volker Runde, *Lectures on Amenability*, Lectures Notes in Mathematics 1774, Springer Verlag, 2002.

[3] Francis Edward Su, *The Banach-Tarski Paradox*, Minor thesis, part of PhD, Harvard University, 1990.

[4] Robert M. French *The Banach-Tarski Theorem*[2] , The Mathematical Intelligencer, vol. 10, no. 4, Springer-Verlag New York, 1988.

[1] The standard text on the subject. It contains (almost) everything you ever will want to know about the paradox. It also contains lot of open paradox-related problems.

[2] A popular introduction to the *ideas* behind the proof of the theorem. This is where I found the notion of “shifting towards infinity”, as mentioned in the introduction.